(I) **Introduction**

(A) **Classification of Loadings**

Characteristics of the dynamic problem:

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(B) **Mathematical Modeling**

It provides the link between the real physical system and the mathematically feasible solution.

(i) **DEGREE OF FREEDOM**

DOF is the number of independent coordinate necessary to specify the configuration or position of a system at any time.
(ii) **Lumped-mass idealization**

Define the displacement or acceleration at those discrete points where the mass of structure is concentrated. Elastic deformations are limited entirely to localized spring elements.

Horizontal stiffness of the frame $k = 2 \times 12 \frac{EI}{h^3}$
A model of SDOF is not adequate
A model of 3-DOFs is much better

**Note:**

**(iii) Generalized coordinate**

*case of uniformly distributed mass

*Assume that the deflected shape is expressed as the sum of a series of specified displacement patterns compatible with the prescribed support conditions, \( \phi_n(x) \),*
Example

\[ \bar{p}(x,t) \theta \]
\[ \bar{m}(x) \]
\[ u(x,t) \]
\[ f_j(x,t) = \bar{m}(x) \ddot{u}(x,t) \]

Use only one term in (2), \( u = Z(t) \varphi(x) \) \(-\text{(3)}\) \( \delta u(x,t) = \delta Z(t) \varphi(x) = \varphi \delta Z \) \(-\text{(3a)}\)

\( \delta W_E = \)

Define \( p^*(t) \equiv \int_0^L \bar{p} \varphi dx \) \(-\text{(4)}\) \( m^* \equiv \int_0^L \bar{m}(x) \varphi^2 dx \) \(-\text{(5)}\)

then \( \delta W_E = \)

On the other hand, \( \delta W_i = \)

Note: \( K \) is curvature \( \sim \) \( u'' = Z \varphi'' \) \( \delta (Z \varphi'') = \varphi'' \delta Z \)

Define \( k^* = \int_0^L EI \varphi''^2 dx \) \(-\text{(6)}\)

then \( \delta W_i = \)

(a) = (b), we have

\[ \delta Z \left[ p^*(t) - m^* \ddot{Z} - k^* Z \right] = 0 \quad \Rightarrow \quad m^* \ddot{Z}(t) + k^* Z(t) = p^*(t) \quad \text{-\text{(7)}} \]

In conclusions, \( *Z(t) \) is called the generalized coordinate, \( m^* \) is called the generalized mass, \( k^* \) is called the generalized stiffness, and \( p^* \) is the generalized or equivalent load.
(iv) Damping

Damping force \( f_d \) removes energy (heat, radiation) from the system.

Physically: (1) ambient:

(2) material:

(3) interface:

The last two kinds are also called structural damping or solid damping.

1. Consider the case that a viscous fluid flows through a slot, around a piston in a cylinder, or around the journal of a bearing. (viscous damping)

2. Experiments indicate that an internal damping is rate-independent, i.e. independent of the cyclic frequency.

3. Coulomb damping

Mathematically

The damping in actual structure is usually represented in a highly idealized manner.

(a) viscous damping

\[ f_d = cu \]  \( -----(8) \)

Energy dissipated in viscous damping is

Consider the case the system is subjected to a sinusoidal excitation, \( p(t) = p_0 \sin(\omega t) \). In the following figure, \( \phi \) is a phase angle.

\[ p_0 \quad p(t) = p_0 \sin(\omega t) \]

\[ u_0 \quad u(t) = u_0 \sin(\omega t - \phi) \]

It is observed that the energy is frequency dependent.
Let us investigate the relationship between damping force and displacement.

Not a single-value function, but a loop, hysteresis loop. The area within the hysteresis loop gives the dissipated energy.

(2) is a nonlinear viscous form.

We can use

\[ f_D + f_S \]

(The shaded area is same as the one in Figure 1. The area contributed by elastic force \( f_s \) is zero!!)
(b) Nonviscous damping (Hysteretic damping)

A mathematical model which has the property of frequency independence is provided by the concept of hysteretic damping. The simplest device that can be used to represent rate-independent linear damping is to assume

\[ \eta = \text{damping coefficient} \]

Interpretation:

If \( u(t) = u_0 \sin \omega t \), then \( f_x = k u_0 \sin \omega t \) \hspace{1cm} (a)

(12) becomes \( f_D = \frac{\eta k}{\omega} \omega u_0 \cos \omega t = \eta k u_0 \cos \omega t \) \hspace{1cm} (b)
(C) Formulation of Motion Equation

Recall the static case, i.e. the determination of equilibrium equation.

(i) d’Alembert’s principle

Every state of motion may be considered at any instant as a state of equilibrium if the inertia forces are taken into account. The concept that a mass develops an inertial force proportional to its acceleration and opposing it is known as d’Alembert’s principle.

\[ f_i + f_D + f_S = p \]

\[ m\ddot{u} + c\dot{u} + ku = p(t) \quad ----(1) \]

Example: (Chopra 8.2)
【solution】

Chose $\theta$ as a coordinate.

$$\sum M_0 = 0$$

$$I_1 \ddot{\theta} + f_{11} \times \frac{L}{2} + I_2 \ddot{\theta} + f_{12} \times \frac{9L}{8} + c \dot{\theta} + f_3 \times \frac{L}{2} - P \times \frac{9L}{8} = 0$$

$$I_1 = \frac{mL^2}{12}, \quad I_2 = \frac{m}{12} \left[ \left( \frac{L}{4} \right)^2 + \left( \frac{L}{4} \right)^2 \right] = \frac{mL^2}{96}$$

$$\frac{mL^2}{12} \dddot{\theta} + \frac{mL}{2} \ddot{\theta} \times \frac{L}{2} + \frac{ml^2}{96} \dddot{\theta} + (9mL \dddot{\theta}) \times \frac{9L}{8} + c \theta + \frac{kL}{2} \theta \times \frac{L}{2} - P \times \frac{9L}{8} = 0$$

$$\left( \frac{103}{64} mL^2 \right) \dddot{\theta} + c \dot{\theta} + \frac{kL^2}{4} \theta = \frac{9L}{8} p(t) \quad m^* k^* p^*(t)$$

Define $\omega^* = \sqrt{\frac{k^*}{m^*}} = \sqrt{\frac{16k}{103m}}$
(ii) Energy Methods

These methods are based on the variation of calculus.

(a) Virtual-work method

It can be applied to a more complex system with the scalar operation.

By applying a virtual displacement, we have

\[ \delta W_e = \delta W_i \quad \delta W_e = \delta U \quad \text{----------(2)} \]

For a rigid body, it is clear that \( \delta W_e = 0 \) \quad (2a)

By applying \( \delta u \), the following equations hold.

\[
\begin{align*}
p(t) \delta u - m \ddot{u} \delta u - c \dot{u} \delta u - k u \delta u & = 0 \\
(p(t) - m \ddot{u} - c \dot{u} - ku) \delta u & = 0 \\
\therefore \delta u & \neq 0 \quad \therefore m \ddot{u} + c \dot{u} + ku = p(t) \quad \text{----------(1)}
\end{align*}
\]

Try the previous example by the same approach.

Example: A rigid massless bar (upper one) and a rigid uniform bar with mass \( m \).
(b) Hamilton’s Principle

Continuous and discrete systems
Scalar operation
Implied inertia and elastic forces

\[ \int_{t_i}^{t_f} \delta(T - U) \, dt + \int_{t_i}^{t_f} \delta W_{nc} \, dt = 0 \]  

(3)

In which,

\( U \) = potential energy, strain energy and potential of any conservative external forces

\( \delta W_{nc} \) = non-conservative work, i.e. damping force and arbitrary external loads
Example:

\[
\begin{align*}
T &= \frac{1}{2}mu^2 \\
U &= \frac{1}{2} ku^2 \\
\delta W_{nc} &= p(t)\delta u - cu\delta u
\end{align*}
\]
Example:

\[ \bar{p}(x,t) \quad u(x,t) = \varphi Z \]

\[ T = \frac{1}{2} \int_{0}^{l} \bar{m} u^2 dx \]

\[ \delta T = \]

On the other hand,

\[ U = \frac{1}{2} \int_{0}^{l} EI u'^2 dx \]

\[ \delta U = \]

\[ \delta W_{nc} = \]

By application of Hamilton’s principle, it is easily seen that

\[ \int_{h}^{l_{h}} (m^* \ddot{Z} - k^* Z \delta Z) dt + \int_{h}^{l_{h}} p^* \delta Z dt = 0 \quad \ldots \ldots (a) \]

note: \[ \int_{h}^{l_{h}} \dot{Z} \delta \dot{Z} dt = -\int_{h}^{l_{h}} \ddot{Z} \delta Z dt \quad \ldots \ldots (b) \]

\[ (b) \rightarrow (a) \Rightarrow -m^* \dddot{Z} - k^* Z + p^* = 0 \]

\[ m^* \dddot{Z} + k^* Z = p^* \]
In a discrete system, various kinds of energy can be expressed as the functions of the generalized coordinate, $q_i$, whose corresponding generalized force is $Q_i$.

$$T = T(q_i, \dot{q}_i) \quad U = U(q_i) \quad \delta W_{nc} = Q_i \delta q_i$$

From Hamilton’s principle,

$$\int_{t_1}^{t_2} (\delta T - \delta U) dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left[ \frac{\partial T}{\partial q_i} \delta q_i + \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i - \frac{\partial U}{\partial q_i} \delta q_i + Q_i \delta q_i \right] dt = 0$$

Example: A SDOF system

$$T = \frac{1}{2} m \ddot{u}^2 \quad U = \frac{1}{2} k u^2 \quad \delta W_{nc} = p(t) \delta u - c \dot{u} \delta u$$

Carrying out differentiation,

$$\frac{\partial T}{\partial \ddot{u}} = m \ddot{u} \quad \frac{\partial T}{\partial u} = 0 \quad \frac{\partial U}{\partial u} = k u$$

Substituting them into (4), we have

$$-0 + m \ddot{u} + ku = p - c \dot{u} \quad i.e. \quad m \ddot{u} + c \dot{u} + ku = p \ldots (1)$$
Example:

【solution】:

\[
\begin{align*}
    x &= l \sin \theta \\
    y &= l \cos \theta \\
    T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} ml^2 \dot{\theta}^2 \\
    U &= mgl (1 - \cos \theta)
\end{align*}
\]

Note that

\[
\frac{\partial T}{\partial \theta} = ml^2 \dot{\theta} \quad \quad \frac{\partial U}{\partial \theta} = mgl \sin \theta
\]
(iii) Support excitation

\[ u'(t) = u_g(t) + u(t) \]  \hspace{1cm} (5)

In above, \( u(t) \) is the relative displacement. By equilibrium, we have

\[
m\ddot{u} + c\dot{u} + ku = 0 \\
m(\ddot{u}_g + \ddot{u}) + c\dot{u} + ku = 0 \\
\Rightarrow m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g
\]  \hspace{1cm} (6)

Note that \(-m\ddot{u}_g\) may be deemed as an effective force.

(iv) Influence of gravity force

Consider a SDOF system. The motion equation is \( m\ddot{u} + ku = p(t) \)

The same system is arranged as following.
The motion equation expressed with reference to the static-equilibrium position is not affected by gravity force.

(D) Solution Techniques (SDOF)

(i) Analytical Approaches

These approaches are mainly for linear systems.

(a) Classical Solution

(b) Duhamel’s Integral
(c) Fourier Integral or Transform

When \( p(t) = 1 \times e^{i\omega t} \), the motion equation can be written as

\[
m\ddot{u} + c\dot{u} + ku = 1 \times e^{i\omega t} \quad \cdots (3)
\]

Assume \( u(t) = H(\omega) \cdot e^{i\omega t} \) \quad \cdots (4)

(4) \rightarrow (3) \quad H(\omega) = \frac{1}{-m\omega^2 + ic\omega + k} \quad \cdots \cdots (5)

(ii) Numerical Approaches
(II) Free Vibration

\[ m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0 \]  \hspace{1cm} (1)

\[ u(t) = Ge^{\lambda t} \]  \hspace{1cm} (2)

(2) \rightarrow (1) \hspace{1cm} \left( m\lambda^2 + c\lambda + k \right) Ge^{\lambda t} = 0 \hspace{1cm} (3)

Define \( \omega_n^2 = k / m \) \hspace{1cm} (4)

\[ \rightarrow \lambda^2 + \frac{c}{m}\lambda + \omega_n^2 = 0 \hspace{1cm} \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega_n^2} \]  \hspace{1cm} (5)

(A) Undamped System \( c = 0 \)

\[ \lambda = \pm i\omega_n \]

\[ u(t) = G_1 e^{i\omega_nt} + G_2 e^{-i\omega_nt} = A \sin \omega_n t + B \cos \omega_n t \] \hspace{1cm} (a)

The period is easily found since \( \omega_n T_n = 2\pi \) and \( T_n = 2\pi / \omega_n \).

The lag time is \( \theta / \omega_n \)

This is a harmonic vibration with circular frequency \( \omega_n \).
Define \( f_n = \frac{1}{T_n} = \frac{\omega_n}{2\pi} \) \( \text{cps or Hertz} \) \( (4) \)

Example

![Diagram of a beam with mass and spring]

The mass of beam can be neglected. Determine the vibration frequency of the mass.

【solution】

(B) Damped System \( c \neq 0 \)

Define \( c_c = 2m\omega_n \) \( (1) \)

There are three conditions, \( c < c_c , \ c = c_c , \ c > c_c \)

(i) \( c < c_c \)

Let \( \xi = c / c_c = c / 2m\omega_n \) \( (2) \)

Therefore, \( \xi < 1 \)

\[ \lambda = -\xi \omega_n \pm i\omega_n \sqrt{1 - \xi^2} \] \( (3) \)

Define \( \omega_D = \omega_n \sqrt{1 - \xi^2} \) \( (4) \)

\( (3) \rightarrow \lambda = -\xi \omega_n \pm i\omega_D \) \( (5) \)
\[ u(t) = G_1 e^{-\xi \omega_D t + i \omega_D t} + G_2 e^{-\xi \omega_D t - i \omega_D t} \]

Example \( \dot{u}(0) = 0, \ u(0) \neq 0 \)

- \( c \rightarrow \) larger
- \( \xi \rightarrow \) larger
- \( \omega_D \rightarrow \) smaller
- \( T_D \rightarrow \) larger
- infinite
(ii) Critically Damped \( ( c = c_c, \, \xi = 1 ) \)

Under this case \( \lambda = -\omega_n \) \( u(t) = (A + B t) e^{-\omega_n t} \) \( \quad (a) \)

\[
u(t) = \{[\dot{u}(0) + \omega_n u(0)] t + u(0) \} e^{-\omega_n t} \quad \ldots (10)\]

Note:
(iii) **Overdamped System** \( (c > c_c, \xi > 1) \)

(C) Logarithmic Decrement of Damping

Define logarithmic decrement as \( \delta = \ln \frac{u_n}{u_{n+1}} \) \hspace{1cm} (1)
\[
\begin{align*}
\frac{u_1}{u_{m+1}} &= \frac{u_1}{u_2} \times \frac{u_2}{u_3} \ldots \ldots \frac{u_m}{u_{m+1}} = e^\delta \times e^\delta \ldots \ldots e^\delta = e^{m\delta} \\
\end{align*}
\]

(4)

Example

\[W = 4000 \text{ lb}, \ k = 10 \text{ k/in}, \ \xi = 5\%
\]

What’s the time taken to reduce the amplitude from 0.1 inch to 0.01 inch?

【solution】
(III) Harmonic Vibration

(A) Dynamic Magnification Factors

\[ m \ddot{u}(t) + c \dot{u}(t) + k u(t) = p_0 \sin \omega t \]  \hspace{1cm} (1)

(i) Undamped System

\[ u = u_c + u_p \]  \hspace{1cm} (2)

\[ u_p = G \sin \omega t \]  \hspace{1cm} (3)

(3) \rightarrow (1) \quad -m\omega^2 G \sin \omega t + kG \sin \omega t = p_0 \sin \omega t \quad \quad G = \frac{p_0}{k} \frac{1}{1-(\omega/\omega_n)^2} \quad (a)

Define frequency ratio \( \beta = \omega/\omega_n \) \hspace{1cm} (4)

From (a), we have \( G = \frac{p_0}{k} \frac{1}{1-\beta^2} \) \hspace{1cm} (5)

Therefore, \( u(t) = \) \hspace{1cm} (6)

Example: given \( u(0) = 0 \), \( \dot{u}(0) = 0 \)

\[ A = \frac{p_0}{k} \frac{-\beta}{1-\beta^2} \quad B = 0 \quad \rightarrow \quad u(t) = \frac{p_0}{k} \frac{1}{1-\beta^2} (-\beta \sin \omega_n t + \sin \omega t) \]  \hspace{1cm} (7)

Note:

(ii) Damped System

From free vibration, \( u_c = e^{-\xi\omega_n t} \rho \cos(\omega_d t - \theta) \)
Let \( u_p = G_1 \sin \omega t + G_2 \cos \omega t \) \hspace{1cm} (3a)

By similar approach, \( u_p = \frac{p_0}{k} \frac{1}{(1-\beta^2)^2 + (2\xi \beta)^2} [(1-\beta^2) \sin \omega t - 2\xi \beta \cos \omega t] \)

First term damps out after the transient period. \( R_d \) is a dynamic magnification factor and a function of frequency ratio and damping ratio. When \( \xi = 0 \), (10) becomes \( R_d = \frac{1}{\left|1 - \beta^2\right|} \) \hspace{1cm} (13)

Diagrams for \( R_d(\beta, \xi) \) and \( \phi(\beta, \xi) \) (Chopra P.78)
(B) Vector Relationship in Harmonic vibration

\[ p(t) = p_0 \sin \omega t \quad u(t) = u_0 \sin(\omega t - \phi) \]  

\[ f_s = ku = ku_0 \sin(\omega t - \phi) = \]  

\[ f_D = c\dot{u} = c\omega u_0 \cos(\omega t - \phi) = \]  

\[ f_I = m\ddot{u} = -m\omega^2 u_0 \sin(\omega t - \phi) = \]  

Consider phase angle, the phase of the response with respect to the force.

(i) \( \beta = 1 \quad \phi = 90^\circ \)

(ii) General Case
If $\beta >> 1 \rightarrow \omega_n$ is small, $\omega$ is large, $\phi \rightarrow \pi$ and $f_i$ is large.

Example:

\[
F(t) = 200 \sin 5.3t \quad lb \\
W = 15 \quad k \\
I = 69.2 \quad in^4 \\
\xi = 5\% \\
\]

Determine the maximum base shear.

\[
\max(f_s(t) + f_D(t))
\]

【solution】
(C) Resonance

\[ R_d = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \]

If \( \beta = 1 \) \( R_d = \frac{1}{2\xi} \)

When \( \xi = 0 \) \( R_d = \infty \)

In fact, \( (R_d)_{\text{max}} \) does not occur at \( \beta = 1 \) if \( \xi \neq 0 \)
To investigate the characteristics of vibration at resonance, we assume that

\[ u(0) = 0 \quad , \quad \dot{u}(0) = 0 \quad , \quad (\beta = 1) \rightarrow \phi = \frac{\pi}{2} \]

\[ u(t) = e^{-\xi \omega_n t} \left( A \sin \omega_d t + B \cos \omega_d t \right) + u_0 \cdot \sin(\omega t - \phi) \]  

(2)

If \( \xi = 0 \)
Note that the first term is infinitive when $t$ is infinitive.

(D) Evaluation of Damping

(i) Equivalent Damping

(ii) Resonance testing

Example
Under harmonic testing, the following is obtained.
When $\omega = \omega_n$, $u_0 = 5 \ "$, and when $\omega = 5\omega_n$, $u_0 = 0.02 \ "$. Determine $\xi$.

【solution】
(iii) Half-Power Method

Determine $\xi$ from the frequency at which the response is reduced to $(\frac{1}{\sqrt{2}})\rho_{\beta-1}$, i.e. the power is half that of resonance.
(E) Vibration Isolation

Although there are a wide variety of applications in regard to the harmonic vibration, to name a few, e.g. unbalance in rotating machine (rotor, disk, shaft) and whirling & critical speed of rotating shaft, herein we only discuss vibration isolation and seismic instruments. The former consists of two problems.

a. Machine’s vibration force to foundation

b. System attached to moving supports (support motion problem)

\[ p(t) = p_0 \sin(\omega t) \]

\[ f_{T} = \frac{\text{max. base force}}{\text{applied force amplitude}} = \frac{(f_T)_0}{p_0} \]  

(1)
(i) Transmissibility

\[ TR_2 = \frac{\text{amplitude of equipment}}{\text{base - motion amplitude}} = \frac{\ddot{u}_0}{\ddot{u}_g} \]  

\[ \ddot{u}_g = \ddot{u}_g \sin \omega t \]

Note: \( \beta \geq \sqrt{2} \) and a small damping.
Example

$I = 120 \text{ in}^4, \quad L = 10 \text{ ft}, \quad W = 3860 \text{ lb}$

Produce a force of 7000 lb with $\omega = 60 \text{ rad/sec}, \quad \xi = 10 \%$. Determine force transmitted to the beam.

(ii) Equipment isolation

\[ \ddot{u}(t) = \beta^2 \dot{u}_g R_d \sin(wt - \phi) \]

\[ \therefore \ddot{u} = \ddot{u}_g + \ddot{u} = \ddot{u}_g \left[ \sin \omega t + \beta^2 R_d \sin(\omega t - \phi) \right] \]

\[ = \ddot{u}_g \left[ \sin \omega t + \beta^2 R_d \cos \phi \sin \omega t - \beta^2 R_d \cos \omega t \sin \phi \right] \]

\[ = \ddot{u}_g \sqrt{(1 + \beta^2 R_d \cos \phi)^2 + (\beta^2 R_d \sin \phi)^2} \sin(\omega t - \alpha) \]

\[ = \ddot{u}_g \sqrt{1 + 2 \beta^2 R_d \cos \phi + \beta^4 R_d^2} \sin(\omega t - \alpha) \quad (a) \]
Example

\[ K = 2135.8 \text{ lb/in} \]
\[ W = 15000 \text{ lb} \]
\[ \omega_n = 7.41 \text{ rad/sec} \]
\[ \xi = 5\% \]
\[ \ddot{u}_g = 5.14667 \sin 5.3 \ t \]

Determine maximum base shear.

Method 1
(E) Measurement Devices

(i) Accelerometer

\[ a = \ddot{u}_g = \ddot{u}_{g_0} \sin \omega t \quad \quad -m\ddot{u}_g = -m\ddot{u}_{g_0} \sin \omega t \]

Therefore, the response is

\[ u(t) = -\frac{m\ddot{u}_{g_0}}{k} R_s \sin(\omega t - \phi) \]

(ii) Displacement Meter

\[ u_g = u_{g_0} \sin \omega t \quad \quad u_{g_0} \text{ is to be determined.} \]
Figure 3.2.7 Deformation, velocity, and acceleration response factors for a damped system excited by harmonic force.

(Figure 1)
(IV) Linear Response – Time Domain

\[ p(t) \rightarrow \text{system} \rightarrow u(t) \]

arbitrary force, excitation \quad \text{response}

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<td>Numerically efficient*</td>
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(A) Unit Impulse Response Function

![Diagram](image)

**Figure 4.1.1** (a) Unit impulse; (b) response to unit impulse.
* How to determine $h(t)$

One is to employ the technique of Laplace transform, another is to use the concept of free-vibration.

$$m\ddot{u} + c\dot{u} + ku = \delta(t - \tau)$$

An alternate method is based on $F\Delta t = m\Delta V$, the relationship between impulse and change of momentum.

$$\rho = \left\{ \frac{\hat{h}(0) + h(0)}{\omega_d} \right\}^2 + h^2(0)\right\}^2 = \quad \text{and} \quad \theta = \tan^{-1}\left( \frac{\hat{h}(0) + h(0)}{\omega_d, h(0)} \right) = \quad \hat{h}(0) + h(0)$$
(B) Duhamel Integral

Recall

\[ u(t) = \int_0^t p(\tau) h(t - \tau) d\tau \]

When \( h(t) \) is obtained, we have

\[ u(t) = \frac{1}{m \omega_n} \int_0^t p(\tau) e^{-\frac{\xi}{\omega_n}(t - \tau)} \sin \omega_n (t - \tau) d\tau \]  \hspace{1cm} (1)

Example

(a) \( t \leq t_d \)

\[ u(t) = \frac{1}{m \omega_n} \int_0^t p_0 \sin \omega_n (t - \tau) d\tau = \]  \hspace{1cm} (2)

(b) \( t > t_d \)

\[ u(t) = \frac{1}{m \omega_n} \int_0^t p(t) \sin \omega_n (t - \tau) d\tau = \]  \hspace{1cm} (3)
(C) Response Spectrum

A response spectrum is a plot of maximum “response” (e.g. displacement, stress, acceleration, etc.) of SDOF systems as a function of some system parameters (usually, natural frequency or period).

Example 1

\[ u(t) = \frac{p_0}{k} (1 - \cos \omega_n t) = \frac{p_0}{k} (1 - \cos \frac{2\pi}{T_n} t) \]

\[ R_d = |1 - \cos \frac{2\pi}{T_n} t|_{\text{max}} \quad (1) \]

\[ u(t) = \frac{p_0}{k} (\cos \omega_n t \cos \omega_d t_d + \sin \omega_n t \sin \omega_d t_d - \cos \omega_n t) \]

\[ t \leq t_d \quad \text{Use (2) in page 41.} \]

\[ t > t_d \quad \text{Use (3) in page 41.} \]
Conclusion:

Example 2

\[
\begin{array}{|c|c|c|}
\hline
\frac{t_d}{T_n} & R_d(t \leq t_d) & R_d(t > t_d) \\
\hline
0 & 0 & 0 \\
0.125 & 0.293 & 0.76 \\
0.25 & 1 & 1.414 \\
0.375 & 1.707 & 1.85 \\
0.5 & 2 & 2 \\
\vdots & 2 & \vdots \\
\hline
\end{array}
\]

Let \( \dot{\alpha}(t) = 0 \) to get \( u_{\text{max}} \).

\[
\dot{u} = \frac{p_0}{k} \left( -\frac{1}{t_d} + \omega_n \sin \omega_n t + \frac{\cos \omega_n t}{t_d} \right) = 0
\]
$t > t_d$ Use (3) in page 43 by letting $t = t_d$.

$$u(t_d) = \frac{p_0}{k} \left( -\cos \omega_n t_d + \frac{\sin \omega_n t_d}{\omega_n t_d} \right)$$

$$\dot{u}(t_d) = \frac{p_0}{k} \left( -\frac{1}{t_d} + \omega_n \cdot \sin \omega_n t_d + \frac{\cos \omega_n t_d}{t_d} \right)$$

Recall the response of free vibration, which is $u(t) = u(0) \cdot \cos \omega_n t + \frac{\dot{u}(0)}{\omega_n} \cdot \sin \omega_n t$.

Example 3

Determine the max blast force that can be sustained if the displacement is to be limited to 5mm

if (1) $t_d = 0.4$ s (2) $t_d = 0.04$ s
In short duration, large part of applied load is resisted by inertia of the structure, and the stresses produced are much smaller.

Note:
(V) Frequency Domain Analysis

(A) Fourier Series

\[ m \ddot{u}(t) + c \dot{u}(t) + ku(t) = p_0 \sin \omega t \quad \text{or} \quad p_0 \cos \omega t \quad (1) \]

The responses are respectively,

\[ u(t) = \frac{p_0}{k} \left[ R_d^2 \left(1 - \beta^2 \right) \sin \omega t - 2 \xi \beta \cos \omega t \right] \quad (2) \]

\[ u(t) = \frac{p_0}{k} \left[ R_d^2 \left(1 - \beta^2 \right) \cos \omega t + 2 \xi \beta \sin \omega t \right] \quad (2a) \]

As a result,

\[ u(t) = \frac{a_0}{k} + \frac{R_d^2}{k} \sum_{j=1}^{\infty} \left[ a_j (1 - \beta_j^2) \cos \omega_j t + \ a_j 2 \xi \beta_j \sin \omega_j t + \ b_j (1 - \beta_j^2) \sin \omega_j t - \ b_j 2 \xi \beta_j \cos \omega_j t \right] \quad (5) \]

Example

Determine the steady-state response under the periodic load.

\[ a_0 = \frac{1}{T_p} \int_0^{T_p} p(t) \, dt = 0 \]

\[ a_j = \frac{2}{T_p} \int_0^{T_p} p(t) \cos \left(2 \pi j t / T_p \right) \, dt = 0 \]

\[ b_j = \frac{2}{T_p} \int_0^{T_p} p(t) \sin \left(2 \pi j t / T_p \right) \, dt = 4 \frac{p_0}{j \pi} \]

if \( j \) is odd
(II) (B) Complex Response Function

(i) F.S. in complex form

\[ p(t) = \sum_{\infty}^{\infty} C_j \exp(i \omega_j t) \quad (1) \]

\[ \text{Where} \quad C_j = \frac{1}{T_p} \int_0^{T_p} p(t) \exp(-i \omega_j t) \ dt \quad (2) \]

Note 1:

The physical meaning of \( H( \cdot ) \) is an output measure (displacement) for unit input (force).

☆Proof
Conclusion:

Example

\[ C_j = \frac{1}{T_p} \int_{0}^{T_p} p(t) \exp(-i \omega_j t) \, dt \]

\[ = \frac{1}{T_p} \int_{0}^{T_p/2} p_0 \exp(-i \omega_j t) \, dt + \frac{1}{T_p} \int_{T_p/2}^{T_p} -p_0 \exp(-i \omega_j t) \, dt \]

\[ = -\frac{2p_0 i}{j \pi} \quad \text{odd} \quad = 0 \quad \text{even} \]

\[ H(\omega_j) = 1/(-m \omega_j^2 + k) = 1/k(1-\beta_j^2) \]

As a result,

\[ u(t) = \]
Note 2:

Steady-state motion will be given by either $\text{Re}(u)$ or $\text{Im}(u)$, depending on whether the excitation is $\cos \omega t$ or $\sin \omega t$.

Proof

Note 3: Relationship between $C_j$ and $a_j$ and $b_j$:

Note: $C_{-j} = C^*_j$ \hspace{1cm} (7)

$$p(t) = \sum_{-\infty}^{\infty} C_j \exp(i \omega_j t) =$$
Note 4: Discretization of \( p(t) \) and \( C_j \)

\[
C_j = \frac{1}{T_p} \int_0^{T_p} p(t) \exp(-i \omega_j t) \, dt \quad \Delta \omega = \omega_1 = 2\pi / T_p \quad \omega_j = j \omega_1 \quad (2)
\]

\[
= \frac{1}{N \Delta t} \sum_{n=0}^{N-1} p(t_n) \exp(-i \omega_j t_n) \Delta t \quad t_n = n \Delta t
\]

\[
= \frac{1}{N} \sum_{n=0}^{N-1} p_n \exp(-i 2\pi n j / N) \quad j = 0, 1, 2, \ldots, N - 1 \quad (2*)
\]

Remark:

\[
p(t) = \sum_{-\infty}^{\infty} C_j \exp(i \omega_j t)
\]

\[
p(t_n) = \sum_{-\infty}^{\infty} C_j \exp(i \omega_j t_n) = \sum_{j=-M}^{M} C_j \exp(i 2\pi n j / N)
\]

\[
p_n = \sum_{j=0}^{N-1} C_j \exp(i 2\pi n j / N) \quad j = 0, 1, 2, \ldots, N - 1 \quad (\text{Let } 2M + 1 = N) \quad (1*)
\]

Remark:

To save numerical labor, the Cooley-Tukey algorithm for the fast Fourier transform (FFT) in 1965 can be employed. The number of complex products for the original FFT algorithm is given by \((N/2) \log_2 N\). For example, if \(N = 1024 = 2^{10}\), the FFT algorithm requires 0.5% of the computational effort necessary in standard computation.
(ii) Fourier Integral

Necessary condition of existence is that \( \int_{-\infty}^{\infty} |p(t)| dt \) must be finite.

\[
p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\omega) \exp(i\omega t) d\omega
\]

(10)

\[
C(\omega) = \int_{-\infty}^{\infty} p(\tau) \exp(-i\omega \tau) d\tau
\]

(11)

Example

Given \( p(t) = \delta(t) \), determine \( u(t) \).

\[
C(\omega) = \int_{-\infty}^{\infty} p(\tau) \exp(-i\omega \tau) d\tau = \]

\[
\begin{align*}
  u(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{m(-\omega^2 + 2i\xi \omega + \omega_n^2)} \cdot \exp(i\omega t) d\omega \\
  &= \frac{-1}{2\pi m \omega_n^2} \int_{-\omega_n}^{\omega_n} \frac{1}{(\beta^2 - 2i\xi \beta - 1)} \cdot \exp(i\omega t) d\omega \\
  &= \frac{-1}{2\pi m \omega_n^2} \int_{-\omega_n}^{\omega_n} \frac{1}{(\beta - \alpha_1)(\beta - \alpha_2)} \cdot \exp(i\omega t) d\omega =
\end{align*}
\]
(iii) Computer implementation

The duration of excitation is $t_d$ and a longer duration $t_f$ is chosen. We have a finite Fourier transform. Therefore, from Eq.(11), we have

$$C(\omega) = \int_{-\infty}^{\infty} p(\tau) \exp(-i\omega \tau) d\tau = \int_{0}^{t_f} p(\tau) \exp(-i\omega \tau) d\tau$$

From Eq.(12), it yields

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \cdot C(\omega) \cdot \exp(i\omega t) d\omega = \frac{1}{2\pi} \sum_{j} H(\omega_j) \cdot C(\omega_j) \cdot \exp(i\omega_j t) \cdot \Delta \omega$$

Summary:
(C) Relationship between $h(t)$ and $H(\omega)$

From Eq.(12) in page 51,

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \cdot C(\omega) \cdot \exp(i\omega t) d\omega$$

Use Eq.(11) in page 51,

$$u(t) =$$
(VI) Numerical Methods

Accuracy
Convergence:
Stability:
Computer implementation:

Three types of time-stepping procedures are:

(A) Interpolation of Loading

Only for linear system, convenient for SDOF systems
Response $u_{r+1}$ is consisted of the responses from (a) free vibration due to $u_i$ and $\dot{u}_i$, (b) force vibration by step force $p_i$, and (c) force vibration by $\Delta p_i \tau / \Delta t_i$
(B) Finite Difference Method

(i) Central Difference Method

This method solves 2nd order DE directly.

At time “t”,

\[ m\ddot{u}_t + c\dot{u}_t + ku_t = p_t \]  \hspace{1cm} (1)

The Taylor expansion of \[ u_{t+\Delta t} \] about \[ u_t \] leads to

\[ u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \frac{\ddot{u}_t}{2!} (\Delta t)^2 + \frac{\dddot{u}_t}{3!} (\Delta t)^3 + ... \]  \hspace{1cm} (2)

Substituting Eq.(4) and Eq.(5) into Eq.(1), we have

\[ m \left( \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2} \right) + c \left( \frac{u_{i+1} - u_{i-1}}{2\Delta t} \right) + ku_i = p_i \]

\[ \left[ \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t} \right] u_{i+1} = p_i + \left( \frac{2m}{(\Delta t)^2} - k_i \right) u_i + \left( \frac{-m}{(\Delta t)^2} + \frac{c_i}{2\Delta t} \right) u_{i-1} \]  \hspace{1cm} (6)
2. This method is an explicit integration method, i.e. new response values depend only on quantities obtained in the preceding steps. In Eq.(6), it doesn’t involve \( k_i \), promising when nonlinear case. The restoring force appears explicitly as it depends only on the response at time “i”, not on the unknown response at “i+1”.

Example

\[
m = 0.1 \quad k/\text{in} \cdot s^2 \\
c = 0.2 \quad k/\text{in} \cdot s \\
k = 5 \quad k/\text{in} \\
\dot{u}_0 = \ddot{u}_0 = 0
\]

\[p_0 = 0, \quad p_{0.1} = 5 \quad k, \quad p_{0.2} = 8, \quad p_{0.3} = 7, \quad p_{0.4} = 5, \quad p_{0.5} = 3, \quad p_{0.6} = 2, \quad p_{0.7} = 1, \quad p_{0.8} = 0\]

\[\Delta t = 0.025\]

\[
\frac{m}{\Delta t^2} + \frac{c}{2\Delta t} = 164 \\
u_0 = \dot{u}_0 = 0 \quad \text{and} \quad \ddot{u}_0 = 0 \\
u_{-1} = 0
\]

\[i = 0 \quad t_1 = 0.025 \quad 164u_1 = P_0 + 0 + 0 \quad \Rightarrow \quad u_1 = 0\]

\[i = 1 \quad t_2 = 0.050 \quad 164u_2 = 1.25 + (320 - 5)u_1 + (-160 + 4)u_0
\]

\[= 1.25 + 315u_1 - 156u_0 = 1.25 \quad \Rightarrow \quad u_2 = 0.0076\]
(C) Nowmark β-Method

\[ m \ddot{u}_i + c_i \dot{u}_i + k_i u_i = p_i \quad (1) \]

\[ m \ddot{u}_{i+1} + c_{i+1} \dot{u}_{i+1} + k_{i+1} u_{i+1} = p_{i+1} \quad (2) \]

\[ (2) - (1), \text{ we have} \quad m \Delta \ddot{u}_i + c_i \Delta \dot{u}_i + k_i \Delta u_i = \Delta p_i \quad (3) \]

Note: \( \Delta u_i = u_{i+1} - u_i \), \( \Delta \dot{u}_i = \dot{u}_{i+1} - \dot{u}_i \), \( \Delta u_i = u_{i+1} + u_i \)

Assume that damping and stiffness properties remain constant during the time step. In fact, the secant stiffness can be used and be modified by total equilibrium.

\[ \dot{u}_i = \dot{u}_0 + \int_0^{\Delta t} \ddot{u}(\tau) \, d\tau \quad \text{can be expressed as} \]
(i) Constant average acceleration

\[ \ddot{u}_t = \frac{1}{2} (\ddot{u}_i + \ddot{u}_{i+1}) \]  
\[ \dot{u}_t = \dot{u}_i + \frac{1}{2} (\ddot{u}_i + \ddot{u}_{i+1}) \tau \]

Let \( \tau = \Delta t \),

\[ \Delta \dot{u}_j = \frac{1}{2} (\ddot{u}_i + \ddot{u}_{i+1}) \Delta t \]

By integration of (a),

By integration of (b),

\[ u_j = u_i + \dot{u}_i \tau + \frac{1}{4} (\ddot{u}_i + \ddot{u}_{i+1}) \tau^2 \]
This is an implicit integration method. Converting such a formulation to an explicit form is highly desirable.

\( \dot{u}_i = \ddot{u}_i + \frac{\Delta \ddot{u}_i}{\Delta t} \tau \) (a*)

Integration of (a*) \( \Rightarrow \dot{u}_\tau = \dot{u}_i + \ddot{u}_i \tau + \frac{\Delta \ddot{u}_i \tau^2}{2} \) (b*)

Let \( \tau = \Delta t \), Eq.(b*) can be rewritten as \( \Delta \dot{u}_i = \dot{u}_i \Delta t + \Delta \ddot{u}_i \frac{\Delta t}{2} \) (6a)

\( = \frac{1}{2} (\dddot{u}_i + \dddot{u}_{i+1}) \Delta t \) (6*)
Note:

We use the linear acceleration method as an example to present an explicit iteration procedure. The key point is to express $\Delta \dddot{u}_i$,  $\Delta \dddot{u}_i$ in terms of  $\Delta u_i$ and then solve $\Delta u_i$.

\[(7a) => \Delta \dddot{u}_i = \frac{6}{\Delta t^2} \Delta u_i - \frac{6}{\Delta t} \dddot{u}_i - 3\dddot{u}_i \]

\[(6a) => \Delta \dddot{u}_i = \dddot{u}_i \Delta t + \Delta \dddot{u}_i \frac{\Delta t}{2} + \left(\frac{6}{\Delta t^2} \Delta u_i - \frac{6}{\Delta t} \dddot{u}_i - 3\dddot{u}_i \right) \frac{\Delta t}{2} = \frac{3}{\Delta t} \Delta u_i - 3\dddot{u}_i - \dddot{u}_i \frac{\Delta t}{2} \]

Substituting Eqs.(8) and (9) into Eq.(3), it yields

\[m \left[ \frac{6}{\Delta t^2} \Delta u_i - \frac{6}{\Delta t} \dddot{u}_i - 3\dddot{u}_i \right] + c_i \left[ \frac{3}{\Delta t} \Delta u_i - 3\dddot{u}_i - \dddot{u}_i \frac{\Delta t}{2} \right] + k_i \Delta u_i = \Delta p_i \]

\[\left[ \frac{6}{\Delta t^2} m + \frac{3}{\Delta t} c_i + k_i \right] \Delta u_i = \Delta p_i + m \left[ \frac{6}{\Delta t} \dddot{u}_i + 3\dddot{u}_i \right] + c_i \left[ 3\dddot{u}_i + \dddot{u}_i \frac{\Delta t}{2} \right] \]

Finally,  $u_{i+1} = u_i + \Delta u_i$
Example

\[ m = 0.1 \text{ k/in.s}\^2 \]
\[ c = 0.2 \text{ k/in.s} \]
\[ k = 5 \text{ k/in} \]
\[ u_0 = \dot{u}_0 = 0 \]

\[ p_0 = 0, \quad p_{0.1} = 5 \text{ k}, \quad p_{0.2} = 8, \quad p_{0.3} = 7, \quad p_{0.4} = 5, \quad p_{0.5} = 3, \quad p_{0.6} = 2, \quad p_{0.7} = 1, \quad p_{0.8} = 0 \]

**Solution**

\[ \Delta t = 0.1 \text{ sec is used.} \]

Eq.(10) \rightarrow \( \frac{6}{\Delta t^2} m + \frac{3}{\Delta t} c + k_i = 66 + k_i \quad k_i = 5 \text{ or zero.} \)

\[ \Delta p_i + \left( \frac{6m}{\Delta t} + 3c \right) \frac{\Delta u_i}{\Delta t} + \left( \frac{3m + c}{\Delta t / 2} \right) \frac{\Delta u_i}{\Delta t} = \Delta p_i + 6.6 \dot{u}_i + 0.31 \ddot{u}_i \]

Eq.(9) \rightarrow \[ \Delta \dot{u}_i = 30 \Delta u_i - 3 \dot{u}_i - 0.05 \ddot{u}_i \]

\[ t_0 : \quad u_0 = 0 \quad \dot{u}_0 = 0 \quad \ddot{u}_0 = \frac{1}{m}(p_0 - c_0 \dot{u}_0 - k_0 u_0) = 0 \]

\[ t_1 : \quad i = 0 \quad 71 \Delta u_0 = (5 - 0) + 6.6 \dot{u}_0 + 0.31 \ddot{u}_0 = 5 \quad \Delta u_0 = 0.07 < 1.2 \text{ OK} \]

\[ u_i = 0.07 \quad \text{(compared with 0.073)} \]
(D) Wilson $\theta$-method

Consider the equilibrium at $t + \theta \Delta t$.

\[
\begin{array}{ccc}
  & i & \quad & i+1 \quad & j \\
  i & t & \quad & t + \Delta t & \quad & t + \theta \Delta t \\
\end{array}
\]

\[m \Delta \ddot{u}_j + c_i \Delta \dot{u}_j + k_i \Delta u_j = \Delta p_j \quad \quad (1)\]

But now $\Delta \dot{u}_i = u_j - u_i = u_{i+\theta \Delta t} - u_i$ (note $\Delta u_i = u_{i+1} - u_i = u_{i+\Delta t} - u_i$)

Let $\Delta t$ in Eq.(C-10) be replaced by $\theta \Delta t$.

\[
\left[ \frac{6}{(\theta \Delta t)^2} m + \frac{3}{\theta \Delta t} c_i + k_i \right] \Delta \ddot{u}_i = \Delta \ddot{p}_i + m \left[ \frac{6}{\theta \Delta t} \dot{u}_i + 3 \ddot{u}_i \right] + c_i \left[ 3 \dot{u}_i + \ddot{u}_i \frac{\theta \Delta t}{2} \right] \quad \quad (2)
\]

Once $\Delta \dot{u}_i$ is obtained, Eq.(C-8) $\rightarrow$ $\Delta \ddot{u}_i = \frac{6}{(\theta \Delta t)^2} \Delta \ddot{u}_i - \frac{6}{\theta \Delta t} \dot{u}_i - 3 \ddot{u}_i \quad \quad (3)$
A discussion on stability:

\[ u_{t+\Delta t} = [A] \cdot u_t + [B] \cdot p_{t+\Delta t} \]

\[ u_{t+2\Delta t} = [A] \cdot u_{t+\Delta t} + [B] \cdot p_{t+2\Delta t} \]

\[ = [A] \cdot ([A] \cdot u_t + [B] \cdot p_{t+\Delta t}) + [B] \cdot p_{t+2\Delta t} \]

\[ = [A]^2 \cdot u_t + [A] \cdot [B] \cdot p_{t+\Delta t} + [B] \cdot p_{t+2\Delta t} \]

\[ u_{t+n\Delta t} = [A]^n \cdot u_t + [A]^{n-1} \cdot [B] \cdot p_{t+n\Delta t} + [A]^{n-2} \cdot [B]^2 \cdot p_{t+2\Delta t} + \cdots + [B] \cdot p_{t+n\Delta t} \]

Let all load are zeros. \[ u_{t+n\Delta t} = [A]^n \cdot u_t \]

Check whether \[ [A]^n \] is bounded. It is found that the requirement is \[ \theta > 1.37 \].

Example: Use \( \theta = 1.5 \)

\[ [30.67 + k] \cdot \Delta \hat{u}_i = \Delta \hat{p}_i + 4.6 \hat{u}_i + 0.315 \hat{u}_i \]

\[ t_0 : u_0 = 0 \quad \hat{u}_0 = 0 \quad \hat{u}_0 = \frac{1}{m} (p_0 - c_0 \hat{u}_0 - k_0 u_0) = 0 \]

\[ \hat{t}_1 : i = 0 \quad 35.67 \quad \Delta \hat{u}_0 = (\frac{5+8}{2} - 0) + 0 + 0 \quad \Delta \hat{u}_0 = 0.182 \]

(4) \( \rightarrow \Delta \hat{u}_0 = \frac{1}{1.5} \left[ \frac{6}{(1.5\Delta t)^2} \times 0.182 - 0 - 0 \right] = 32.4 \)

(5) \( \rightarrow \Delta \hat{u}_0 = 0 + 32.4 \times \frac{0.1}{2} = 1.62 \quad \hat{u}_1 = 0 + \Delta \hat{u}_0 = 1.62 \) (compare with 2.11)

(6) \( \rightarrow \Delta u_0 = 0 + 0 + 32.4 \times \frac{0.1}{6} \times \frac{0.1^2}{6} = 0.054 \quad u_0 = 0.054 \) (compare with 0.070)

(7) \( \rightarrow \hat{u}_1 = \frac{1}{m} (5 - 0.2 \times 1.62 - 5 \times 0.054) = 44.06 \) (compare with 42.3)
(VII) Rayleigh’s Method

(A) Distributed-mass system

Concept: the energy in a free vibrating system must remain constant if no damping forces act to absorb it.

Consider a generalized SDOF system.

\[ u = u_0 \sin \omega_n t \]

\[ T = \frac{1}{2} m u^2 = \]

\[ U = \frac{1}{2} k u^2 = \]

Kinetic energy and potential energy at different time instants:

<table>
<thead>
<tr>
<th></th>
<th>( T )</th>
<th>( U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 = \frac{T_n}{4} = \frac{\pi}{2\omega_n} )</td>
<td>0</td>
<td>( ku_0^2 / 2 ) (i.e. ( U_{\text{max}} ))</td>
</tr>
<tr>
<td>( t_2 = \frac{T_n}{2} = \frac{\pi}{\omega_n} )</td>
<td>( m\omega_n^2 u_0^2 / 2 ) (i.e. ( T_{\text{max}} ))</td>
<td>0</td>
</tr>
</tbody>
</table>

But, \( T_1 + U_1 = T_2 + U_2 \rightarrow U_{\text{max}} = T_{\text{max}} \) \hspace{1cm} (1)

Example 1: Use the deflection under self-load to estimate the fundamental frequency.

Assume that \( u(x,t) = \phi(x) \sin \omega_n t \)

\[ U = \frac{1}{2} \int_0^L EI (u'')^2 \, dx = \]

\[ T = \frac{1}{2} \int_0^L m u'^2 \, dx = \]
Example 2: Also a concentrated mass M is placed at the middle.

Assume that \( u(x,t) = \phi(x) \sin \omega_n t \).

Deflection due to P at the middle is \( \Delta = \frac{PL^3}{48EI} \). Therefore, let

\[
\phi(x) = \Delta \left( \frac{3x}{L} - 4\left(\frac{x}{L}\right)^3 \right) \quad x \leq L/2
\]

\[
T_{\text{max}} = (T_h)_{\text{max}} + (T_M)_{\text{max}} = 2 \cdot \frac{1}{2} \int_0^{L/2} m \phi^2 \omega_n^2 \, dx + \frac{1}{2} M \Delta^2 = \int_0^{L/2} m \phi^2 \omega_n^2 \, dx + \frac{1}{2} M [\phi(L/2)\omega_n]^2
\]

\[
= 0.24285 mL \omega_n^2 \Delta^2 + \frac{1}{2} M \Delta^2 \omega_n^2 \quad (a)
\]

\[
U_{\text{max}} = \frac{1}{2} \cdot 2 \int_0^{L/2} EI \phi''^2 \, dx = \frac{1}{2} P \Delta = \frac{1}{2} \left( \frac{48EI}{L^3} \Delta \right) \Delta \quad (b)
\]

(a) = (b) \quad \Rightarrow \quad \omega_n = \frac{48EI}{(0.4857 mL + M) L^3}

Note: Owing to the difficulty in assuming deflection curves for higher modes, this method is mainly used for 1st mode.
(B) Lumped-Mass System

\[ u(x,t) = \phi(x) \sin \omega_n t \quad \text{and} \quad \dot{u}(x,t) = \omega_n \phi(x) \cos \omega_n t \]

\[ T = \frac{1}{2} \sum_{i=1}^{3} m_i \dot{u}_i^2(x,t) = \frac{1}{2} \sum_{i=1}^{3} m_i \omega_n^2 \phi_i^2(x) \cos^2 \omega_n t \]

\[ U = \frac{1}{2} \sum_{i=1}^{3} k_i [u_i(x,t) - u_{i+1}(x,t)]^2 = \frac{1}{2} \sum_{i=1}^{3} k_i [\phi_i(x) - \phi_{i+1}(x)]^2 (\sin^2 \omega_n t) \]

Example 1: Use an arbitrary deflection curve.

A three DOF system is given, with \( k_1 = 600 \text{ kips/in} \), \( k_2 = 1200 \text{ kips/in} \), \( k_3 = 1800 \text{ kips/in} \),

\( m_1 = 1 \text{ kips/in \cdot sec}^2 \), \( m_2 = 1.5 \text{ kips/in \cdot sec}^2 \), \( m_3 = 2 \text{ kips/in \cdot sec}^2 \).

Determine the fundamental frequency.

【solution】
Example 2: Use the static deflection curve of own weight as the shape.

<table>
<thead>
<tr>
<th></th>
<th>Spring elongation $\Delta \phi_i$</th>
<th>Mass deflection $\phi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\frac{4.5g}{1800} = \frac{9g}{3600}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2.5g}{1200} = \frac{7.5g}{3600}$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\frac{g}{600} = \frac{6g}{3600}$</td>
<td></td>
</tr>
</tbody>
</table>

Omitting the term $\frac{g}{3600}$,

(C) Improved Rayleigh’s Method

Use the static deflection curve induced by inertia force, which is $m\ddot{u}_{\text{max}} = ma^2u_{\text{max}}$ as the shape.
When Eq.(1*) is used, we have (inertia force is deemed as $W_i$ now)

$$
\omega_n^2 = \frac{\sum_{i=1}^n W_i \phi_i}{\sum_{i=1}^n m_i \phi_i^2} = \frac{22.5 \times \frac{203.625}{1800} + 24.75 \times \frac{136.125}{1800} + 18 \times \frac{65.25}{1800}}{1 \times \left(\frac{203.625}{1800}\right)^2 + 1.5 \times \left(\frac{136.125}{1800}\right)^2 + 2 \times \left(\frac{65.25}{1800}\right)^2} = 211.194 = \text{above}
$$

$\omega_n = 14.533$
(VIII) MDOF Systems

(A) Direct Formulation of Motion Equation

Column stiffness = $I/2$

\[
\begin{bmatrix}
    m_1 & 0 & 0 \\
    0 & m_2 & 0 \\
    0 & 0 & m_3
\end{bmatrix}
\begin{bmatrix}
    \ddot{u}_1 \\
    \ddot{u}_2 \\
    \ddot{u}_3
\end{bmatrix}
\begin{bmatrix}
    k_1 & -k_1 & 0 \\
    -k_1 & k_1 + k_2 & -k_2 \\
    0 & -k_2 & k_2 + k_3
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3
\end{bmatrix}
= \begin{bmatrix}
    p_1 \\
    p_2 \\
    p_3
\end{bmatrix}
\]

\[m \ddot{u} + ku = p \tag{1}\]

(B) FEM Formulation

(i) Influence coefficient

Dynamic equilibrium at any time instant

\[f_I + f_D + f_S = p \tag{1}\]
Elastic force matrix

\[
\mathbf{f} = \begin{bmatrix}
    f_{s1} \\
    f_{s2} \\
    \vdots \\
    f_{sn}
\end{bmatrix}_{N \times 1}
\]

Denote \( k_{ij} \) as the elastic force developed at \( i \) due to a unit displacement at \( j \). Then,

\[
f_{s1} = k_{i1}u_1 + k_{i2}u_2 + \cdots + k_{iN}u_N \\
f_j = ku
\]  

(2)

Similarly, denote \( c_{ij} \) as the damping force developed at \( i \) due to a unit velocity at \( j \). Then,

\[
f_{d1} = c_{i1}\dot{u}_1 + c_{i2}\dot{u}_2 + \cdots + c_{iN}\dot{u}_N \\
f_d = c\dot{u}
\]  

(3)

Finally, let \( m_{ij} \) be the inertia force developed at \( i \) due to a unit acceleration at \( j \) and have

\[
f_i = m\ddot{u}
\]  

(4)

Eqs.(2)(3) & (4) → Eq.(1)

\[
m\ddot{u} + c\dot{u} + k\dot{u} = p
\]  

(5)

Example 1: Use concept of influence coefficient to determine \( k_{ij} \).

【Solution】
\[
\begin{pmatrix}
k_1 & -k_1 & 0 \\
-k_1 & k_1 + k_2 & -k_2 \\
0 & -k_2 & k_2 + k_3
\end{pmatrix}
= \frac{12EI}{L^3}
\begin{pmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}
\]

\(\text{(ii) stiffness matrix}\)

First, consider the shape functions related with the axial direction.

Let \(u(x) = ax + b\)

By this equation and boundary condition, \(x = 0, u_1 = b\) and \(x = L, u_4 = aL + b\)

Next, determine \(\phi_2\).

\[
EI y'' = M(x) = Bx - C \quad \quad EI y = \frac{B}{6}x^3 - \frac{1}{2}C x^2 + Dx + F
\]  
(a)
Similarly, \( \phi_3 = x(1 - \frac{x}{L})^2 \quad \phi_5 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \quad \phi_6 = \frac{x^2}{L}\left(\frac{x}{L} - 1\right) \) \hspace{1cm} (8)

Finally, use shape functions to determine \( k_{ij} \). Use the determination of \( k_{14} \) as an example.

Example 2: Determination of \( k_{23} \)

The force developed at 2 due to a unit rotation at 3, \( u_3 = 1 \).
Actual: displacement field is $u(x) = \phi_3 \cdot u_3 = \phi_3 \cdot 1 = \phi_3$, the associated internal moment is

$$M(x) = EI u''(x) = EI \phi_3''$$

$$k = \begin{bmatrix}
\frac{AL^2}{2I} & 0 & 0 & -\frac{AL^2}{2I} & 0 & 0 \\
6 & 3L & 0 & -6 & 3L \\
2L^2 & 0 & -3L & L^2 \\
\frac{AL^2}{2I} & 0 & 0 \\
sym. & 6 & -3L \\
& 2L^2 \\
\end{bmatrix} \frac{2EI}{L^3} \quad (10)$$

(iii) Mass matrix

Consistent mass matrix

Determine $m_{14}$. 
Therefore, in general,

\[ m_{ij} = \int_0^L m(x) \phi_i \phi_j \, dx \quad i, j = 1 \sim 6 \quad (11) \]

\[
\begin{bmatrix}
140 & 0 & 0 & 70 & 0 & 0 \\
156 & 22L & 0 & 54 & -13L \\
4L^2 & 0 & 13L & -3L^2 \\
140 & 0 & 0 \\
\text{sym.} & 156 & -22L \\
4L^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\bar{m}L \\
420
\end{bmatrix} \quad (12)
\]

Example 3: Determine mass matrix \( m \). Each element has a unit length.
By superposition,

\[
\begin{bmatrix}
4 & 5 & 6 & 7 & 8 & 9 \\
280 & 0 & 0 & 140 & 0 & 0 \\
312 & 44 & 0 & 108 & -26 & \\
8 & 0 & 26 & -6 & \\
280 & 0 & 0 & \\
312 & -44 & \\
8 & 
\end{bmatrix}
\]

\[
m_2 =
\]

The effective mass matrix \( m \) is delineated in the box.

**Lumped mass matrix**

Assumption: the structure is lumped at the nodal coordinates where the translational displacement is defined.
\[
\begin{bmatrix}
\bar{m}L/2 \\
\bar{m}L/2 \\
0 \\
\bar{m}L/2 \\
\bar{m}L/2 \\
0
\end{bmatrix}
\]

Example 3A: Determine mass matrix \( \bar{m} \) by the lumped-mass approach.

\[
m_1 = \begin{bmatrix}
210 & 0 & 210 & 210 \\
0 & 210 & 210 & 0
\end{bmatrix} \quad \quad m_2 = \begin{bmatrix}
420 & 0 & 420 & 420 \\
0 & 420 & 420 & 0
\end{bmatrix}
\]

\[
m_1 + m_2 = \begin{bmatrix}
210 & 0 & 210 & 210 & 630 & 630 & 0 & 420 & 420 \\
0 & 630 & 630 & 0 & 420 & 420 & 0
\end{bmatrix}
\]

Example 3B: Determine mass matrix \( m \) by the lumped-mass approach and neglecting the axial deformation.

\[
\begin{bmatrix}
630 & 0 & 630 & 0 & 420 & 420 \\
0 & 420 & 420 & 0
\end{bmatrix}
\]
Example 4: Determine mass matrix $m$ by the lumped-mass approach and considering the axial deformation. The mass density for all members is the same, $\bar{m}$.

$$m = \begin{bmatrix} 5 & 6 & 8 & 9 \\ 630 & 0 & 420 & 0 \end{bmatrix}$$

Example 5

$$m = \begin{pmatrix} 3m & 0 \\ 0 & m \end{pmatrix}_{2 \times 2}$$
Example 6: A mass of M is placed at the node.

Consistent mass is more general. Comparison of two masses is made as follows.

<table>
<thead>
<tr>
<th></th>
<th>Lumped</th>
<th>Consistent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Simple</td>
<td></td>
</tr>
<tr>
<td>Computation effort</td>
<td>Less (static condensation)</td>
<td></td>
</tr>
<tr>
<td>Matrix type</td>
<td>Diagonal* Some zeroes</td>
<td></td>
</tr>
<tr>
<td>Accuracy</td>
<td>Only translational DOF</td>
<td></td>
</tr>
</tbody>
</table>

(iv) Direct Stiffness Method

The stiffness of the complete structure can be obtained by merely adding the element stiffness coefficients properly.

The relationship between the local coordinates and the global coordinates (denoted with an asterisk) are as follows.

\[
\begin{align*}
    u_1 &= u_1^* \cos \theta + u_2^* \sin \theta \\
    u_2 &= -u_1^* \sin \theta + u_2^* \cos \theta \\
    u_3 &= u_3^*
\end{align*}
\]
Let $\lambda = \cos \theta$ $\mu = \sin \theta$, we have

\[
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    \vdots \\
\end{bmatrix}
= 
\begin{bmatrix}
    \lambda & \mu & 0 \\
    -\mu & \lambda & 0 \\
    0 & 0 & 1 \\
    \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
    u_1^* \\
    u_2^* \\
    u_3^* \\
    \vdots \\
\end{bmatrix}
\]

i.e. $u = Bu^*$ \hspace{1cm} (14)

Example 7: Determine $k_{12}^*$
Example 8: Determine the stiffness matrix of the shear building by the DSM.

\[
k^* = B^T k B = \frac{2EI}{L^3}
\]

\[
\begin{bmatrix}
6 & 0 & 3L & -6 & 0 & 3L \\
\frac{AL^2}{2I} & 0 & 0 & -\frac{AL^2}{2I} & 0 \\
2L^2 & -3L & 0 & L^2 \\
6 & 0 & -3L & \\
\frac{AL^2}{2I} & 0 & \\
2L^2
\end{bmatrix}
\]

\[
k^*_a + k^*_b + k^*_c = \frac{2EI}{L^3}
\]

\[
\begin{bmatrix}
6 & 0 & 3L & -6 & 0 & 3L \\
0 & \frac{AL^2}{2I} & 0 & 0 & -\frac{AL^2}{2I} & 0 \\
3L & 0 & 2L^2 & -3L & 0 & L^2 \\
-6 & 0 & -3L & 12 & 0 & 0 & -6 & 0 & 3L \\
0 & -\frac{AL^2}{2I} & 0 & 0 & \frac{AL^2}{I} & 0 \\
3L & 0 & L^2 & 0 & 0 & 4L^2 & -3L & 0 & L^2 \\
-6 & 0 & -3L & 12 & 0 & 0 \\
0 & -\frac{AL^2}{2I} & 0 & 0 & \frac{AL^2}{I} & 0 \\
3L & 0 & L^2 & 0 & 0 & 4L^2
\end{bmatrix}
\]
Example 9: Determine $k$ and $m$ of the following frame, if the axial deformation of all elements is neglected. A horizontal force of $p$ is applied at Node 3.

\[
4EI, 1.5A, 2L, 1.5\bar{m}
\]

DOFs at node 2 are 1, 2, and 3, horizontal, vertical, and rotational, respectively. DOFs at node 3 are 4, 5, and 6, horizontal, vertical, and rotational, respectively.

Element stiffness matrices for elements 21 & 34 (same) and element 23 are:

\[
\begin{bmatrix}
6 & 0 & 3L & -6 & 0 & 3L \\
0 & AL^2/2I & 0 & 0 & -AL^2/2I & 0 \\
3L & 0 & 2L^2 & -3L & 0 & L^2 \\
-6 & 0 & -3L & 6 & 0 & -3L \\
0 & -AL^2/2I & 0 & 0 & AL^2/2I & 0 \\
3L & 0 & L^2 & -3L & 0 & 2L^2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
AL^2/2I & 1.5 & 0 & 0 & -AL^2/2I & 1.5 & 0 & 0 \\
6 \times \frac{1}{2} & 3L & 0 & -6 \times \frac{1}{2} & 3L & 2L^2 \times 2 & 0 & -3L & L^2 \times 2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{AL^2}{2I} \times 1.5 & \frac{AL^2}{2I} & 0 & 0 & \frac{AL^2}{2I} \times 1.5 & 0 & 0 \\
6 \times \frac{1}{2} & -3L & 2L^2 \times 2 & 0 & 0 & \frac{AL^2}{2I} & 0 & 0 \\
\end{bmatrix}
\]

sym. \[
\begin{bmatrix}
6 \times \frac{1}{2} & -3L & 2L^2 \times 2 \\
\end{bmatrix}
\]
\[
\begin{align*}
\begin{bmatrix}
\frac{3AL^2}{8I} + 6 & 0 & 3L & -\frac{3AL^2}{8I} & 0 & 0 \\
3 + \frac{AL^2}{2I} & 3L & 0 & -3 & 3L \\
\frac{2EI}{L^3} & 4L^2 + 2L^2 & 0 & -3L & 2L^2 \\
\frac{3AL^2}{8I} + 6 & 0 & 3L \\
sym. & 3 + \frac{AL^2}{2I} & -3L & 6L^2
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
m_a^* &= B^T m_a B = \frac{mL}{420} \\
&= \begin{bmatrix}
156 & 0 & 22L & \ldots \ldots \\
0 & 140 & 0 & \ldots \ldots \\
22L & 0 & 4L^2 & \ldots \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{bmatrix} \\
m_c^* &= m_a^* \\
\end{align*}
\]

\[
\begin{align*}
m_b^* &= \frac{\bar{m} L}{420} \\
&= \begin{bmatrix}
140 \times 3 & 0 & 0 & 70 \times 3 & 0 & 0 \\
0 & 156 \times 3 & 22L \times 6 & 0 & 54 \times 3 & -13L \times 6 \\
0 & 132L & 4L^2 \times 12 & 0 & 13L \times 6 & -3L^2 \times 12 \\
210 & 0 & 0 & 140 \times 3 & 0 & 0 \\
0 & 162 & 78L & 0 & 156 \times 3 & -22L \times 6 \\
0 & -78L & -36L^2 & 0 & -132L & 4L^2 \times 12
\end{bmatrix}
\end{align*}
\]
Express the dependent (secondary) DOF in terms of the remaining independent (primary) DOF. The relation between the dependent and the independent DOF is found by establishing the static relation between them.

The system equation in Example 9 is rearranged as

\[
\sum m = \frac{\bar{m} L}{420} = \begin{bmatrix}
576 & 0 & 22L & 210 & 0 & 0 \\
0 & 608 & 132L & 0 & 162 & -78L \\
22L & 132L & 52L^2 & 0 & 78L & -36L^2 \\
210 & 0 & 0 & 576 & 0 & 22L \\
0 & 162 & 78L & 0 & 608 & -132L \\
0 & -78L & -36L^2 & 22L & -132L & 52L^2
\end{bmatrix}
\]
Example 9A: Condensation of the motion equation in Example 9.

$4EI, 1.5A, 2L, 1.5\bar{m}$

\[ T = -k_{ss}^{-1}k_{sp} = \begin{bmatrix} \frac{6L^2}{2L^2} & 2L^3 \\ 2L^3 & 6L^2 \end{bmatrix}^{-1} \begin{bmatrix} 2EI \\ 3L \end{bmatrix} \frac{2EI}{L^3} \begin{bmatrix} 3L \\ 2EI \end{bmatrix} \frac{1}{8L} \begin{bmatrix} -3 \\ -3 \end{bmatrix} \]

Eq.(20)\[ \hat{p}_p = p - 0 = p \]

\[ \hat{k}_{pp} = \frac{2EI}{L^3} (12) + \frac{2EI}{L^3} [3L \ 3L] T = \frac{19.5EI}{L^3} \]

\[ 4\bar{m}L \ddot{u}_1 + \frac{19.5EI}{L^3} u_i = p \]
Alternate Approach

Eqs.(17) and (19) are rewritten in a matrix form.

\[
\begin{bmatrix}
0 & 0 \\
0 & m_{pp}
\end{bmatrix} \begin{bmatrix}
\ddot{u}_s \\
\ddot{u}_p
\end{bmatrix} + \begin{bmatrix}
I & -T \\
0 & \hat{k}_{pp}
\end{bmatrix} \begin{bmatrix}
u_s \\
u_p
\end{bmatrix} = \begin{bmatrix}k_{ss}^{-1}p_s \\
p_p - k_{pp}k_{ss}^{-1}p_s\end{bmatrix}
\]

Recall the original equation, Eq.(16),

\[
\begin{bmatrix}
0 & 0 \\
0 & m_{pp}
\end{bmatrix} \begin{bmatrix}
\ddot{u}_s \\
\ddot{u}_p
\end{bmatrix} + \begin{bmatrix}
k_{ss} & k_{sp} \\
k_{ps} & k_{pp}
\end{bmatrix} \begin{bmatrix}
u_s \\
u_p
\end{bmatrix} = \begin{bmatrix}p_s \\
p_p\end{bmatrix}.
\]

An inverse in $\hat{k}_{pp}$ is not necessary. We use Gauss-Jordan elimination to convert

\[
\begin{bmatrix}
k_{ss} & k_{sp} \\
k_{ps} & k_{pp}
\end{bmatrix}
\]

into a form of

\[
\begin{bmatrix}
I & -\hat{T} \\
0 & \hat{k}_{pp}
\end{bmatrix}.
\]

Then $\hat{T}$, $\hat{k}_{pp}$, and $\hat{p}_p$ are obtained automatically.

Example 10: Use static condensation to reduce the first coordinate.
A note on reduced stiffness matrix $\hat{k}_{pp}$

Theorem: $\hat{k}_{pp} = t^T k t$ (21) in which $t = \begin{bmatrix} T \\ I \end{bmatrix}$ (22)

It can be shown that whether $p_s$ is zero or not does not alter $\hat{k}_{pp}$. Thus, the relationship of $u_s$ and $u_p$, $u_s = T u_p + k_{ss}^{-1} p_s$, can be simplified as $u_s = T u_p$. Accordingly,

$$\begin{bmatrix} u_s \\ u_p \end{bmatrix} = \begin{bmatrix} T \\ I \end{bmatrix} u_p = t u_p$$ (23)
Example 9B:

\[
\begin{bmatrix}
T \\
L
\end{bmatrix} = \begin{bmatrix}
-\frac{3}{8}L \\
-\frac{3}{8}L \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
T^T L \\
L 
\end{bmatrix} = \begin{bmatrix}
-\frac{3}{8}L & -\frac{3}{8}L & 1
\end{bmatrix}
\begin{bmatrix}
\frac{2EI}{L^3} \\
\frac{6L^2}{2L^2} & \frac{2L^2}{6L^2} & \frac{3L}{3L} \\
\frac{1}{3L} & \frac{1}{3L} & \frac{1}{12}
\end{bmatrix}
\begin{bmatrix}
-\frac{3}{8}L \\
-\frac{3}{8}L \\
1
\end{bmatrix} = 19.5 \frac{EI}{L^3}
\]

Example 10A:

\[
\begin{bmatrix}
\frac{1}{2} \\
1
\end{bmatrix}
\begin{bmatrix}
\frac{2}{2} & -\frac{k}{2} & -\frac{k}{2} \\
-\frac{k}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{2} \\
1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} \\
1
\end{bmatrix} = \frac{k}{2}
\]

as before

Interpretation:

Express $PE$, $KE$, and the virtual work of the damping forces in terms of the independent coordinates $u_p$. 

Hence, $\hat{k}$, $\hat{m}$, and $\hat{c}$ may be regarded, respectively, as the stiffness, mass, and damping matrices of the structure corresponding to the independent coordinates. In other word, the same transformation based on static condensation for the reduction of stiffness matrix is also used in reducing the mass and damping matrices.

(vi) *Loading Matrix*

If a load is not placed on the nodes, use a consistent load or a lumped load.

**Consistent load**

$$p(x,t)$$

$$p_i = \int_0^L p(x,t) \cdot \phi_i(t) \, dx$$ (26)

Example11: The floor of a frame is subjected to a load of $p(x,t) = A \cdot \sin(\omega t)$ kN/m.

Determine the consistent load matrix.
(C) Solution Techniques

1. Classic modal analysis – linear, classical

2. Direct Analysis (\( c \) is required) – linear & nonlinear, classical & non-classical

\[ p(x,t) = A \cdot \sin(\omega t) \text{ kN/m} \]
(IX) Free Vibration of MDOF Systems

(A) Natural Frequency and modal shape

Undamped motion equation

\[ m \ddot{u} + k u = 0 \]  \hspace{1cm} (1)

Assume \( u = a \sin(\omega t + \theta) \) \hspace{1cm} (2) \hspace{1cm} \ddot{u} = -\omega^2 u \hspace{1cm} (3)

(2) (3) \hypeq (1)

\[ m\left[-\omega^2 a \sin(\omega t + \theta)\right] + k a \sin(\omega t + \theta) = 0 \hspace{1cm} (-m\omega^2 + k)a = 0 \]  \hspace{1cm} (4)

To get a nontrivial solution \( \rightarrow \| -m\omega^2 + k \| = 0 \) \hspace{1cm} (5)

Example 1

\[ m_1 = 136 \text{ lb} \cdot \text{sec}^2/\text{in} \hspace{0.5cm} k_1 = 30700 \text{ lb} \cdot /\text{in} \]

\[ m_2 = 66 \text{ lb} \cdot \text{sec}^2/\text{in} \hspace{0.5cm} k_2 = 44300 \text{ lb} \cdot /\text{in} \]

Determine the natural frequencies and modal shapes.
Find \( q \) by recalling Eq.(4), i.e.

\[
\begin{bmatrix}
-m_1 \omega^2 + k_1 + k_2 & -k_2 \\
-k_2 & -m_2 \omega^2 + k_2
\end{bmatrix}
\begin{bmatrix}
a_1 \\ a_2
\end{bmatrix} = 0
\]

(4a)

For \( \omega_1 = 11.83 \text{ rad/sec} \)

\[
\begin{bmatrix}
-136 \times 140 + 75000 & -44300 \\
-44300 & -66 \times 140 + 44300
\end{bmatrix}
\begin{bmatrix}
a_1 \\ a_2
\end{bmatrix} = 0
\]

By the first row, \( 55960 a_1 - 44300 a_2 = 0 \). If \( a_1 = 1 \), then \( a_2 = 1.263 \).

This is first modal shape (normal mode shape), \( \phi_1 = \begin{pmatrix} 1.000 \\ 1.263 \end{pmatrix} \)

Similarly, for \( \omega_2 = 32.90 \text{ rad/sec} \), we have \( \phi_2 = \begin{pmatrix} 1.000 \\ -1.629 \end{pmatrix} \)
Example 2: Normalization of mode shapes.

\[
\begin{bmatrix}
1 & 1.263 \\
66 & 1.263
\end{bmatrix}
\begin{bmatrix}
136 \\
66
\end{bmatrix} = 241.31 \\
\phi_1 = \begin{bmatrix}
0.06437 \\
0.08130
\end{bmatrix} \\
\phi_2 = \begin{bmatrix}
0.0567 \\
-0.0924
\end{bmatrix}
\]

Check \( \phi_1^T m \phi_1 = 1 \)

Define \( \phi = (\phi_1, \phi_2, \ldots, \phi_n) \) = modal matrix of the system. \hspace{1cm} (7)

Example 3: Perform a static condensation, in which \( u_2 \) is deemed as the primary coordinate.
Dynamic condensation

Solution of Response

The solution of Eq.(1a) is given by the superposition of the modal harmonic vibrations.

\[
\begin{align*}
\mathbf{u} &= a \sin(\omega t + \theta) \\
& \rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = C_1 \phi_1 \sin(\omega_1 t + \theta_1) + C_2 \phi_2 \sin(\omega_2 t + \theta_2) \\
&= \phi_1 [C_1 \sin \omega_1 t \cos \theta_1 + C_1 \cos \omega_1 t \sin \theta_1] + \phi_2 [C_2 \sin \omega_2 t \cos \theta_2 + C_2 \cos \omega_2 t \sin \theta_2] \\
&= \phi_1 [B_1 \sin \omega_1 t + A_1 \cos \omega_1 t] + \phi_2 [B_2 \sin \omega_2 t + A_2 \cos \omega_2 t] = \sum_{n=1}^{2} \phi_n [A_n \cos \omega_n t + B_n \sin \omega_n t]
\end{align*}
\]
(B) Orthogonality

\[ m\ddot{u} + k u = 0 \quad \rightarrow \quad -\omega^2 m u + k u = 0 \quad \rightarrow \quad -\omega_j^2 m \phi_j + k \phi_j = 0 \quad \omega_j^2 m \phi_j = k \phi_j \quad (1) \]

Example:

\[
\begin{bmatrix}
0.06437 & 0.08130 \\
66 & 136
\end{bmatrix}
\begin{bmatrix}
136 \\
66
\end{bmatrix}
\begin{bmatrix}
0.0567 \\
-0.0924
\end{bmatrix}
= 0
\]

Furthermore, \( \phi_i^T \omega_j \phi_j = \phi_i^T k \phi_j \). \( i \neq j \), \( 0 = \phi_i^T k \phi_j \). \( (3) \)

Generalization
(C) Response Based on Modal Vibrations

\[ u(t) = \sum_{n=1}^{N} \phi_n (A_n \cos \omega_n t + B_n \sin \omega_n t) \]  \hspace{1cm} (1)

Concept of modal expansion:

Denote \( q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t \)  \hspace{1cm} (2)

Based on modal expansion of displacement, Eq.(1) is rewritten as \( u = \sum \phi_i q_i \)

It is analogous to the free vibration response of SDF system. Therefore, \( q_n(t) \) is named as modal coordinate or normal coordinate.
Eq.(4) indeed has a set of $N$ independent equations, i.e. $M_i \ddot{q}_i + K_i q_i = 0$ (5)

In general, $u(t)$ is not harmonic.
Example 5: Determine the free-vibration response if \( u(0) = \phi_1 \), \( \dot{u}(0) = 0 \), \( \phi_1 = \begin{pmatrix} a \\ b \end{pmatrix} \).

\[
\begin{align*}
u &= \sum_{1}^{2} \phi_n \ q_n(t) \\
&= \sum_{1}^{2} \phi_n \ (q_n(0) \cos \omega_n t + \frac{\dot{q}_n(0)}{\omega_n} \sin \omega_n t) \\
q_1(0) &= b
\end{align*}
\]

(2)

Note: A characteristic deflected shape is called a natural mode of vibration of an MDOF system.

Example 6: Determine the free-vibration response if \( u(0) = \phi_2 \), \( \dot{u}(0) = 0 \), \( \phi_2 = \begin{pmatrix} -c \\ d \end{pmatrix} \).
(E) Damped System

\[ \dot{m} \ddot{u} + c \dot{u} + k u = 0 \]  

(1)

We already have

\[
 M = \begin{pmatrix}
 M_1 & 0 \\
 M_2 & \ddots \\
 0 & M_N
\end{pmatrix}
\]

\[
 K = \begin{pmatrix}
 K_1 & 0 \\
 K_2 & \ddots \\
 0 & K_N
\end{pmatrix}
\]
Since \( C_n = \phi_n^T \Sigma \phi_n \), we let \( \xi_n = C_n / 2 M_n \omega_n \)

\[
q_n(t) = e^{-\xi_n \omega_n t} \left[ A_n \cos(\omega_n D t) + B_n \sin(\omega_n D t) \right]
\]

\[
= e^{-\xi_n \omega_n t} \left[ q_n(0) \cos(\omega_n D t) + \frac{\dot{q}_n(0) + \xi_n \omega_n q_n(0)}{\omega_n D} \sin(\omega_n D t) \right]
\]

(3)

Example 7: Determine the free-vibration response if \( u(0) = \phi_1 \) \( \dot{u}(0) = 0 \).

(F) Formulation of Damping Matrix

(i) Classical Damping

Method 1

\[
\zeta = a_0 m + a_1 k
\]

(1)
Example 8: Determine $\zeta$ if $\omega_n = 12.57, 34.33, \text{ and } 46.89 \text{ rad/sec}$. $\xi_1 = \xi_2 = 0.05.$

$$\phi_1 = \begin{pmatrix} 0.401 \\ 0.695 \\ 0.803 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0.803 \\ 0.000 \\ -0.803 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} 0.401 \\ -0.695 \\ 0.803 \end{pmatrix}$$

Eq.(3) $\rightarrow$ 

\[
\begin{pmatrix} 0.05 \\ 0.05 \end{pmatrix} = \begin{pmatrix} 1/12.57 \\ 1/34.33 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}
\]

$a_0 = 0.9198, \quad a_1 = 0.0021$

\[
m = \frac{1}{386} \begin{pmatrix} 400 & 400 \\ 400 & 200 \end{pmatrix}, \quad k = 610 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 1 \end{pmatrix}
\]
Note: Generalization

Method 2

Matrix operation → $c = m \left[ \sum_{n=1}^{N} \phi_n \times \xi_n \times \frac{2 \xi_n \omega_n}{M_n} \right] m$ (5)

It needs only the damping ratios of each mode.

Example:

$\phi_1 = \begin{pmatrix} 0.401 \\ 0.695 \\ 0.803 \end{pmatrix}$, $\phi_2 = \begin{pmatrix} 0.803 \\ 0.000 \\ -0.803 \end{pmatrix}$, $\phi_3 = \begin{pmatrix} 0.401 \\ -0.695 \\ 0.803 \end{pmatrix}$, use $\xi_1 = \xi_2 = 0.05$ and $\xi_3 = 5.93\%$ (same as the above problem)

$c = m \left[ \phi_1 \phi_1^T \frac{2 \xi_1 \omega_1}{M_1} + \phi_2 \phi_2^T \frac{2 \xi_2 \omega_1}{M_2} + \phi_3 \phi_3^T \frac{2 \xi_3 \omega_1}{M_3} \right] m$
(ii) *Nonclassical Damping*

System consists of two or more parts with significantly different level of damping. For instance,

- soil-structure problem ~ nuclear containment 15~20% 3~5%
- fluid-structure problem~ dam fluid damping is negligible

with energy dissipating devices or base isolation.
(X) Force Vibration of MDOF System

\[ m \ddot{u} + c \dot{u} + k u = p \quad (1) \]

\[ u = \phi_1 Y_1 + \phi_2 Y_2 + \cdots = \phi Y \quad (2) \]

(2)→(1) and multiplying \( \phi_n^T \) on both sides, we have

(A) Modal Participation Factor

\[ p = \gamma p(t) \quad \gamma : \text{spatial variation} \quad (4) \]

Define \( \frac{\phi_n^T s}{M_n} \equiv \Gamma_n \) (modal participation factor) \quad (5)

Then

\[ \frac{\phi_n^T p}{M_n} = \frac{\phi_n^T s}{M_n} p(t) = \Gamma_n p(t), \text{ Eq.(3) becomes} \]

\[ \ddot{Y}_n + 2 \xi_n \omega_n \dot{Y}_n + \omega_n^2 Y_n = \Gamma_n p(t) \quad (6) \]

Consider
(B) Mode- Superposition Method

\[ Y_n = \frac{\ddot{Y}_n(0)}{\omega_n} \sin \omega_n t + Y_n(0) \cos \omega_n t + \frac{1}{M_n \omega_n} \int_0^t P_n(\tau) \sin \omega_n (t - \tau) d\tau \]  \hspace{1cm} (1) 

Example

\[ p = \left( \begin{array}{c} 10000 \\ 20000 \end{array} \right) \cdot p(t), \quad \omega_1 = 11.83, \quad \phi_1 = \left( \begin{array}{c} 0.06437 \\ 0.08130 \end{array} \right) \quad \omega_2 = 32.90, \quad \phi_2 = \left( \begin{array}{c} 0.0567 \\ -0.0924 \end{array} \right) \]
Solve the modal equations when \( t \leq 0.1 \).

\[
Y_1 = \frac{2270}{139.24} (1 - \cos \omega_1 t) + \frac{2270}{139.24 \times 0.1} \left( \frac{\sin \omega_1 t}{\omega_1} - t \right)
\]

\[
Y_2 = \quad \text{(expression not shown)}
\]

\[
u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \phi_1 Y_1 + \phi_2 Y_2 = \begin{pmatrix} 0.06437 \\ 0.08130 \end{pmatrix} \left\{ \frac{2270}{139.24} (1 - \cos \omega_1 t) + \frac{2270}{139.24 \times 0.1} \left( \frac{\sin \omega_1 t}{\omega_1} - t \right) \right\}
\]

For \( t \geq 0.1 \) (Free Vibration)

\[
u = \begin{pmatrix} 0.06437 \\ 0.08130 \end{pmatrix} \left\{ \frac{2270}{139.24 \times \omega_1 \times 0.1} \left[ \sin \omega_1 t - \sin \omega_1 (t - 0.1) \right] - \frac{2270}{139.24 \times \omega_1} \cos \omega_1 t \right\}
\]
Approximation Method

Recall \( u_1 = \phi_{11} Y_1 + \phi_{12} Y_2 \) \quad \quad \quad \quad \quad \quad u_2 = \phi_{21} Y_1 + \phi_{22} Y_2

Absolute maximum (ABS)

\[
(u_1)_{\text{max}} = \left| \phi_{11} (Y_1)_{\text{max}} \right| + \left| \phi_{12} (Y_2)_{\text{max}} \right| \quad \quad \quad (u_2)_{\text{max}} = \left| \phi_{21} (Y_1)_{\text{max}} \right| + \left| \phi_{22} (Y_2)_{\text{max}} \right| \tag{3}
\]

Square Root of Sum Squares (SRSS)

\[
(u_1)_{\text{max}} = \sqrt{\left[ \phi_{11} (Y_1)_{\text{max}} \right]^2 + \left[ \phi_{12} (Y_2)_{\text{max}} \right]^2} \quad \quad \quad (u_2)_{\text{max}} = \sqrt{\left[ \phi_{21} (Y_1)_{\text{max}} \right]^2 + \left[ \phi_{22} (Y_2)_{\text{max}} \right]^2} \tag{4}
\]
Based on Eq.(4), the approximate maximum displacements by SRSS are

(C) Response Spectrum Analysis

(i) Effective Seismic Load

\[ m \ddot{u} + c \dot{u} + k u = -m \ddot{u}_g \quad \ddot{u} + 2\xi_n \omega_n \dot{u} + \omega_n^2 u = -\ddot{u}_g \]  \hspace{1cm} (1)

\[ u(t) = \frac{1}{m\omega_n} \int_0^t -m\ddot{u}_g e^{-\xi_n \omega_n (t-\tau)} \sin \omega_n (t-\tau) d\tau \]  \hspace{1cm} (2)

\[ u(t) = \frac{1}{\omega} \int_0^t -\ddot{u}_g e^{-\xi \omega (t-\tau)} \sin \omega (t-\tau) d\tau \quad \xi << 1 \]  \hspace{1cm} (2a)

Denote \[ U(t) = \int_0^t \ddot{u}_g e^{-\xi \omega (t-\tau)} \sin \omega (t-\tau) d\tau \]  \hspace{1cm} (3)
7. Construct a tripartite response spectrum. See the textbook.

(ii) Maximum Displacement

Example: \( m_1 = 1 \text{ k/in \cdot sec}^2 \), \( m_2 = 1.5 \), \( m_3 = 2 \), \( k_1 = 60 \text{ k/in} \), \( k_2 = 120 \), \( k_3 = 180 \).

In addition, \( S_{v,1} = 1.74 \text{ ft/s} \), \( S_{v,2} = 1.41 \), and \( S_{v,3} = 1.20 \). From \( k - \omega^2 m = 0 \), we have

\[
\omega = \begin{pmatrix} 4.58 \\ 9.82 \\ 14.59 \end{pmatrix} \text{ rad/sec}, \quad \phi = \begin{pmatrix} 1 & 1 & 1 \\ 0.644 & -0.601 & -2.57 \\ 0.3 & -0.676 & 2.47 \end{pmatrix}, \quad M = \phi^T \cdot \phi = \begin{pmatrix} 1.801 \\ 2.455 \\ 23.1 \end{pmatrix}
\]
(iii) Maximum Elastic Forces and Base Shears

\[ f_M = k \mathbf{u} = k \mathbf{\phi} Y \]  \hspace{1cm} (13)

Mode 1

\[ (Y_1)_{max} = \frac{\Gamma_1}{\omega_1} S_{v,1} = \frac{1.425}{4.58} \times 1.74 = 0.5414 \]

\[
\begin{bmatrix}
60 & -60 & 0 \\
-60 & 180 & -120 \\
0 & -120 & 300
\end{bmatrix}
\begin{bmatrix}
1 \\
0.644 \\
0.3
\end{bmatrix}
= \begin{bmatrix}
11.56 \\
10.78 \\
6.89
\end{bmatrix}
\]

(a)
Approximate superposition procedure must be applied directly to the response quantity in question.

(iv) Effective Modal Mass

The elastic force  
\[ f_s = k \ddot{u} = k \ddot{\phi} Y = m \dot{\phi}^2 Y \]

\[ \omega^2 = \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \vdots \\ \omega_n^2 \end{bmatrix} \]
Definition: \( M_j \Gamma_j^2 \) is called effective modal mass.

It reflects the relative contribution to the total base shear due to each mode.

\[
M_1 \Gamma_1^2 = 1.801 \times 1.426^2 = 3.66 \rightarrow 81.5\%
\]

\[
M_2 \Gamma_2^2 = 2.455 \times 0.511^2 = 0.641 \rightarrow 14.3\%
\]

\[
M_3 \Gamma_3^2 = 23.1 \times 0.090^2 = 0.187 \rightarrow 4.2\%
\]