ABSTRACT: Mathematical solutions are obtained for several sedimentary problems featuring semi-infinite alluvial channels evolving under diffusional sediment transport. Moving boundaries are considered at one end of the channels, and represent transitions between alluvial reaches and either bedrock-floored channels or bodies of standing water. Three elementary cases are considered: bedrock-alluvial transitions, lake breaches, and prograding deltas. It is shown that idealized formulations of all three problems share the same mathematical structure and admit exact similarity solutions. Elementary solutions can further be assembled to describe composite profiles. This is illustrated by the case of a natural lake undergoing simultaneous breaching and backfill. For both elementary and composite cases, the explicit solutions clarify the link between alluvial profile evolution and the migration of channel boundaries. For the case of lake breaching, for instance, the outlet channel profile is controlled simultaneously by downwards incision and upstream migration of the channel head. The pace of the resulting water level drawdown in turn affects the form of the backfill deposits upstream of the lake.
INTRODUCTION

Sediment transport in alluvial channels constitutes the most common mode of long range sediment delivery. In many cases, this alluvial transit does not occur continuously from upstream to downstream. Instead, alluvial reaches may co-exist with reaches of two different characters: 1) bedrock exposures, along which sediment may move but where erosion is limited; 2) bodies of standing water, in which sediment motion is suppressed and deposits accumulate. In general, transitions between alluvial and non-alluvial reaches do not remain static: they constitute moving boundaries which may migrate upstream or downstream depending on the conditions. This paper is concerned with the mathematical analysis of such transitions.

For channels of finite length, delta shorelines and bedrock-alluvial transitions were earlier treated as moving boundary problems by Swenson et al. (2000) and Parker and Muto (2003), respectively. In this paper, we focus our attention on semi-infinite channels featuring a single moving boundary: alluvial channels bounded on one end by either a bedrock exposure or standing water, with the other boundary sent to infinity. The alluvial profile is assumed to evolve according to a linear diffusion equation. At the moving boundary, compatibility conditions are then imposed depending on the specific case. As sketched in Figure 1, three different cases are considered: A) a moving sediment edge separating an upstream bedrock basement from a downstream alluvial cover; B) the erosion and upstream migration of a lake breach, due to the drainage of a lake by an alluvial channel; C) the progradation of a fluvial delta into a body of standing water (lake or sea).

Three recent photos of Taiwan rivers are presented in Figure 2 to illustrate the issues involved and their relevance to field conditions. Panel 2A shows a semi-alluvial channel with upper reaches of exposed bedrock and lower reaches covered by an alluvial deposit. A visit to the
same location in November 2005 indicates that the transition between exposed bedrock and alluvial bed has moved upstream by roughly one channel width since this photo was taken in May 2003. Panel 2B shows a recent landslide dam formed across a narrow mountain stream. Upstream of this natural dam, a 1.4 km long temporary lake has formed and started to discharge water past the dam crest (see Chen et al. 2005 for a detailed description). A clear transition is observed between the mirror-like quiescent surface of the lake and the white rapids churning down the downstream face of the dam. During the next typhoon season, this transition is expected to migrate upstream upon incision of the outlet channel by flood waters. According to local people, a single typhoon season was sufficient to wash out a similar landslide dam formed a few years ago in a lower reach of the same river. Panel 2C, finally, shows a small mountain dam with hydroelectric facilities that were recently put out of service due to the nearly complete backfill of the reservoir by a prograding alluvial delta. It took only one year for the delta front to advance from the dashed line marked on the photo to its current oblique position.

In all three cases illustrated on Figure 2, the location of the alluvial boundary and its evolution over time are clearly of engineering significance. Over longer time scales, processes of this kind can also be expected to exert a significant influence on the overall river profiles. For both these reasons, the three phenomena have recently attracted much research attention. Mixed bedrock-alluvial channels have for instance been examined by Hovius et al. (2000), and Whipple and Tucker (2002). Field events involving lake drainage through incising outlet channels have been documented by Waythomas et al. (1996), Brooks and Lawrence (1999) and Blair (2001). Finally the progradation of deltas into standing water has been studied by Swenson et al. (2000), Muto (2001), Kostic and Parker (2003), and Bellal et al. (2003, 2005), among others. Despite their apparent differences, these phenomena share some key underlying features which motivate their common treatment here.
In fact, we aim to show in this paper that the three problems illustrated in Figure 1 share the same mathematical structure. In a rather natural way, their initial and boundary conditions can be made to satisfy a scaling symmetry of the diffusion equation. This in turn leads to the existence of self-similar alluvial profiles, evolving in time without changing their shapes. Exact solutions can then be worked out analytically. The approach adopted for this purpose draws upon the work of Voller et al. (2004) who recently exploited the same symmetry to solve a related problem: the progradation of a fluvial delta starting from a channel of zero length. Our objective is to extend the approach to semi-infinite channels, and use it to treat other problems of geomorphological interest.

GOVERNING EQUATION AND SPECIAL SOLUTIONS
Before treating special cases, consider a generic alluvial channel evolving under the geomorphic action of flowing water. The channel is bounded upstream and downstream by positions $x = s_1$ and $x = s_2$, either one of which can be sent to infinity. The morphodynamic evolution of the channel profile is assumed to be governed by the diffusion equation

$$\frac{\partial z}{\partial t} - D \frac{\partial^2 z}{\partial x^2} = 0, \quad s_1 < x < s_2,$$

(1)

where $z(x,t)$ is the elevation of the channel bed, sought as a function of horizontal coordinate $x$ and time $t$, and $D$ is a constant diffusion coefficient proportional to the water discharge per unit width. Equation (1) is standard and applies to many problems in heat flow and solute transport. In the context of fluvial morphodynamics, it expresses conservation of sediment mass. Seminal works in which this equation has been used to solve sediment motion problems include Begin et al. (1981), Ribberink and van der Sande (1985), and Paola et al. (1992), and a review of applications to sedimentary basin filling can be found in Paola (2000). The equation is most simply derived by assuming steady water discharge, constant channel
width, normal water flow, and sediment transport proportional to stream power. Paola et al. (1992) and Paola (2000) provide derivations for geologic time scales based on less restrictive assumptions.

Throughout this paper, the water profile is approximated in the following manner. Over active alluvial reaches, the depth is assumed to coincide with the normal depth. As discussed in Paola (2000), this approximation applies when channels are examined over length scales large relative to the backwater length. As a further simplification, the normal depth is neglected compared to the typical height variations of interest. In regions of standing water, on the other hand, the water level is assumed horizontal. These approximations are reasonable for large scale problems in steep sloped areas, reproducing the sharp contrast between steep, shallow channel flow and deep, nearly horizontal backwater zones. They may nevertheless lead to local errors in uplands, or even cause global discrepancies if applied to gently sloped lowlands. Despite these restrictions, the above idealization is highly useful because it effectively removes the water profile from the problem. Surface water is reduced to a diffusion agent (in active alluvial reaches) or to a boundary condition (at the edge of a body of standing water).

For fixed boundaries $s_1$ and $s_2$, the linear diffusion equation (1) is well-known to admit general solutions, typically constructed using Green functions or Fourier series. Such diffusion solutions over fixed domains have been applied to sedimentary processes and compared with experiments in Begin et al. (1981), Gill (1983) and Capart et al. (1998). This paper focuses instead on moving boundary problems, for which one of the two positions can evolve over time. Coupling between the channel response and boundary motion makes the problem non-linear, and general solutions can no longer be expected. For particular problems, however, special solutions can be derived by exploiting symmetries of the diffusion equation.
Two known solutions of this kind are illustrated in Figure 3. Figure 3A shows the classical solution of Neumann (see Crank 1984), originally derived for the ‘single-phase’ Stefan problem in which conductive heat transfer in a liquid (water) drives the advance of a melting front into an isothermal solid (ice) at the melting temperature. The solution is displayed here for an evolving alluvial channel: sediment is supplied upstream of a lake of constant depth in such a way that the upstream bed level \( z_0 \) at position \( x = 0 \) is held constant. This leads to channel aggradation and to the advance of a delta into the standing water. The well-known exact solution to this problem has the similarity structure

\[
z(x,t) = f(x/\xi(t))z_0, \quad s_1 = 0 < x < s_2(t) = \lambda \xi(t)
\]

where function \( \xi(t) \) is an evolving length scale, function \( f(\sigma) \) is a dimensionless function of dimensionless argument \( \sigma = x/\xi(t) \), and \( \lambda \) is a dimensionless constant, all chosen to satisfy eq. (1) and its boundary conditions.

Recently, Voller et al. (2004) derived an analytical solution for the different problem illustrated in Figure 3B. Instead of a prescribed sediment level upstream of a lake of constant depth, they considered the case of a prescribed sediment influx upstream of a lake of linearly increasing depth. This problem is characterized by the different similarity structure

\[
z(x,t) = f(x/\xi(t))\xi(t), \quad s_1 = 0 < x < s_2(t) = \lambda \xi(t)
\]

in which both the elevation \( z \) and the spatial coordinate \( x \) are normalized with respect to the evolving length scale \( \xi(t) \). The resulting time sequence of self-similar profiles is obtained by simultaneously rescaling both the \( x \) and \( z \) coordinates, rather than just the \( x \) coordinate as in the classical Neumann solution.

In this work, we show that the same similarity structure applies to semi-infinite channels, bounded on one end by a moving boundary, with the other boundary sent infinitely far upstream or downstream. Actual channels are of course finite in length. Idealizing them as
semi-infinite amounts to examining their behavior over length scales that are short compared to the overall length \( L \) of the channel. It also implies that the profile evolution is examined over time scales that are fast compared to the time \( T \) needed for changes at one end of the channel to be felt at the other end. When the evolution is driven by diffusion, this time \( T \) is of the order of \( L^2 / D \) where \( L \) is again the channel length and \( D \) is the diffusion coefficient. Provided that these conditions are met, we will see that several solutions of geomorphic interest can be derived for semi-infinite channels. Before presenting these solutions, the wider family to which they belong is first examined in the next section from a mathematical perspective.

**SIMILARITY SOLUTIONS**

Following Voller et al. (2004), we focus on special solutions of the diffusion equation (1) having the self-similar structure

\[
z(x,t) = f(x/\xi(t))\xi(t),
\]

where function \( f(\sigma) \) expresses the shape of the profile as a function of dimensionless argument \( \sigma = x/\xi(t) \), and \( \xi(t) \) is a function of time having dimension of length, which governs the pace of the self-similar rescaling of the solutions. Substitution of ansatz (4) into the diffusion equation (1) shows that the function \( \xi(t) \) must be of the form

\[
\xi(t) = 2\sqrt{Dt}
\]

where factor 2 is arbitrary and introduced for later convenience. Both (4) and (5) can be surmised on dimensional grounds. From a mathematical point of view, they constitute a special case of the more general invariant solution (see e.g. Hydon 2000)

\[
u(x,t) = t^{k/2}F(xt^{-1/2}).
\]

For any value of \( k \) (not necessarily equal to an integer), these are exact solutions describing the evolution of a scalar \( u \) subject to the dimensionless diffusion equation.
\[ \partial u / \partial t - \partial^2 u / \partial x^2 = 0. \]

Form (4) corresponds to \( k = 1 \), while the classical Neumann solution corresponds to \( k = 0 \).

The assumed scaling symmetry has various consequences. First, relations (4) and (5) imply

\[
\begin{align*}
\frac{d\xi}{dt} &= \frac{2D}{\xi}, \\
\frac{\partial z}{\partial t} &= \frac{2D}{\xi} \left\{ f'(\sigma) - \frac{x}{\xi} f'(\sigma) \right\}, \\
\frac{\partial z}{\partial x} &= f'(\sigma), \\
\frac{\partial^2 z}{\partial x^2} &= \frac{1}{\xi^2} f''(\sigma),
\end{align*}
\]

(7)

where \( f'(\sigma) \) and \( f''(\sigma) \) respectively denote the first and second derivative of the shape function \( f(\sigma) \). The diffusion equation (1) therefore reduces to the ordinary differential equation

\[ f''(\sigma) + 2f(\sigma) f'(\sigma) - f(\sigma) = 0. \] (8)

This is a second order linear ODE with non-constant coefficients. By inspection, one of its basis solutions is simply

\[ f_1(\sigma) = -\sigma, \] (9)

where the minus sign is introduced for later convenience. Reduction of order can be used to find the second basis solution. This must be of the form

\[ f_2(\sigma) = f_1(\sigma) g(\sigma) = -\sigma g(\sigma) \] (10)

where \( g'(\sigma) = h(\sigma) \) must satisfy the first order ODE

\[ \sigma h'(\sigma) + 2(1 + \sigma^2)h(\sigma) = 0. \] (11)

The solution of this separable equation is found to be

\[ h(\sigma) = \exp(-\sigma^2) / \sigma^2. \] (12)

Integration by parts then leads to the second basis solution

\[ f_2(\sigma) = \exp(-\sigma^2) + \sqrt{\pi} \sigma \text{erf}(\sigma) + C, \] (13)

where \( \text{erf}(\sigma) = 2 / \sqrt{\pi} \int_0^\sigma \exp(-y^2)dy \) and \( C \) is an arbitrary constant of integration to be chosen later (different values simplify different problems). Once a choice for \( C \) has been made, the general solution of ODE (8) can be written
\[ f(\sigma) = -A\sigma + B[\exp(-\sigma^2) + \sqrt{\pi}\sigma \text{erf}(\sigma) + C], \]  
where \( A \) and \( B \) are constants to be determined based on the applied boundary conditions. For this purpose, one will further need the derivative

\[ f'(\sigma) = -A + B\sqrt{\pi} \{\text{erf}(\sigma) + C\}. \]  
The assumed similarity structure also imposes certain restrictions on the boundary and initial conditions that can be applied. First, for boundary conditions of ODE (8) to be specified at given values of ratio \( \sigma = x/\xi(t) \), the boundary positions \( s_1 \) and \( s_2 \) must be of the form

\[ s_1 = \lambda_1\xi(t), \quad s_2 = \lambda_2\xi(t), \]  
where scaling constants \( \lambda_i, \ i=1,2 \) can be either given or unknown (and must then be subject to additional boundary conditions). Form (16) allows non-moving boundary conditions, but only at locations \( s_i = 0 \) or \( s_i \to \pm \infty \). Conversely, any \( \lambda_i \) different from zero or infinity yields a moving boundary. Restrictions on boundary speeds then follow from

\[ \frac{ds_i}{dt} = \lambda_i \frac{d\xi}{dt} = \frac{2D\lambda_i}{\xi} = \frac{2D\lambda_i^2}{s_i}, \]  
which implies that products \( s_i(ds_i/dt) \) must be invariant. Other boundary invariants implied by similarity form (4) and their consequences (7) are

\[ z_i/\xi, \quad z_i/s_i \quad \text{and} \quad (\partial z/\partial x)_i, \]  
where index \( i \) is used to indicate boundary values, i.e. \( z_i = z(s_i,t) \) and \( (\partial z/\partial x)_i = (\partial z/\partial x)_{|_{x=s_i}} \). Simple boundary conditions guaranteed to satisfy the similarity structure can therefore be imposed by setting any one of these invariants equal to a prescribed constant, or by specifying a relationship between two different invariants. Examples will be provided below.
Likewise, initial conditions are subject to certain restrictions. Moving boundary conditions must start from the origin at time $t = 0$, i.e. $s(0) = 0$. If instead either of the boundaries is sent to infinity, profiles of constant slope are required on the corresponding half-domain:

$$z(x,0) = \begin{cases} -S_x, & x < 0 \\ -S_x, & x > 0 \end{cases}$$

(19)

The corresponding boundary condition at infinity must be of the form $(\partial z / \partial x) = -S_1$.

The requirements laid out above still leave much room for choice, allowing for any combination of fixed and moving boundaries, each subject to boundary conditions of different types. No systematic classification will be attempted here. Instead, our objective in the remainder of the paper is to derive detailed solutions for problems involving semi-infinite channels. The three elementary problems illustrated in Figures 1A-C are treated in the next three sections: A) bedrock-alluvial transitions; B) lake breaching by channel incision; C) lake filling by a prograding delta. A composite problem involving a combination of cases B) and C) will be treated in a final section.

**BEDROCK-ALLUVIAL TRANSITIONS**

Consider first a bedrock channel of inclination $S_1$ half-buried under a downstream alluvial cover (see Fig. 1A and the more detailed definition sketch of Fig. 4). The transition between the exposed upstream bedrock and the downstream alluvial channel is located at evolving position $s(t)$. At time $t = 0$, it is assumed that $s = 0$ and that the downstream alluvial cover has constant slope $S_2 < S_1$. A steady sediment supply $Q$ is provided upstream of the bedrock channel. As long as $Q < DS_1$, transport along the bedrock floor is supply-limited, and no deposition occurs. Sediment simply transits along the bedrock until it reaches the upstream edge of the alluvial channel. Clear water conditions can also be examined by setting
\( Q = 0 \) (no upstream sediment supply).

The corresponding mathematical problem is written:

\[
\frac{\partial z}{\partial t} - D \frac{\partial^2 z}{\partial x^2} = 0, \quad s(t) < x < \infty, \tag{20}
\]

\[
z = -S_1 x, \quad \frac{\partial z}{\partial x} = \frac{-Q}{D}, \quad x = s(t), \tag{21, 22}
\]

\[
\frac{\partial z}{\partial x} = -S_2, \quad x \to \infty, \tag{23}
\]

\[
z = -S_2 x, \quad x > 0, \quad t = 0. \tag{24}
\]

Here equation (20) governs the evolution of the alluvial channel profile, and is supplemented by two upstream boundary conditions (21), (22), one downstream asymptote (23) and an initial profile (24). Similarity implies that the position of the moving boundary is given by

\[
s(t) = \lambda \xi(t) = 2\lambda \sqrt{D t} \tag{25}
\]

where scaling constant \( \lambda \) is unknown and must be determined as part of the solution.

The solution for this problem and all other problems treated below are constructed in the same way. For this first problem, the procedure will be shown in detail. Equations (20) to (24) above are first rewritten in terms of ratio \( \sigma = x/\xi(t) \) and similarity function \( f(\sigma) \):

\[
f''(\sigma) + 2\{\sigma f'(\sigma) - f(\sigma)\} = 0, \quad \lambda < \sigma < \infty, \tag{26}
\]

\[
f(\lambda) = -S_1 \lambda, \quad f'(\lambda) = -\frac{Q}{D}, \tag{27, 28}
\]

\[
f'(\infty) = -S_2. \tag{29}
\]

The general solution of ODE (26) was derived in the previous section and is given by

\[
f(\sigma) = -A\sigma + B[\exp(-\sigma^2) + \sqrt{\pi}\sigma\text{erf}(\sigma) + C], \tag{30}
\]

and it is convenient for this problem to choose \( C = -1 \). To determine constants \( A \) and \( B \), boundary conditions (28) and (29) are invoked:

\[
f'(\sigma) = -A + B\sqrt{\pi}\{\text{erf}(\sigma) - 1\} = -\frac{Q}{D} \tag{31}
\]
\[ f'(\infty) = -A = -S_2 \] (32)

from which
\[ A = S_2, \quad B = \frac{S_2 - Q/D}{\sqrt{\pi \{ \text{erf}(\lambda) - 1 \}}}. \] (33)

Finally, the remaining boundary condition (27) can be written
\[
f(\lambda) = -A\lambda + B[\exp(-\lambda^2) + \sqrt{\pi \lambda \{ \text{erf}(\lambda) - 1 \}]} = -S_1\lambda
\] (34)

This leads to the following transcendental equation for unknown \( \lambda \):
\[
\lambda - \omega \frac{\exp(-\lambda^2) + \sqrt{\pi \lambda \{ \text{erf}(\lambda) - 1 \}]}}{\sqrt{\pi \{ \text{erf}(\lambda) - 1 \}}} = 0
\] (35)

where \( \omega = (Q/D - S_2)/(S_1 - S_2) \) depends on the rate of sediment supply compared to the equilibrium transport rates at inclinations \( S_1 \) and \( S_2 \). For a given value of \( \omega \), the unique root \( \lambda \) satisfying equation (35) is easy to compute, using for instance a Newton method.

The resulting sediment profiles are illustrated in Figure 5A for the special case of zero upstream sediment supply \( (Q = 0) \). The results shown are obtained for an alluvial slope set to half the bedrock inclination, i.e. \( S_2/S_1 = 0.5 \), leading to a parameter value \( \omega = -1 \). The corresponding value for root \( \lambda \) is \( \lambda = 0.4328 \). Profiles are given for equally spaced values of similarity variable \( \xi = \sqrt{2Dt} = 0, 1, 2, 3, 4, 5 \), rather than for equally spaced times \( t \), and are plotted in dimensionless form using an arbitrary length scale \( L \). Under zero upstream sediment supply, the clear water flow is erosive as it reaches the alluvial cover. Consequently, the transition gradually moves downstream, exposing new bedrock as time advances. The corresponding sediment elevation profiles are concave, degrading an ever greater extent of the downstream alluvial channel. Contrasting with this behavior, convex profiles associated with overloading are illustrated in Figure 5B. The parameters for this example are \( S_2/S_1 = 0.2 \), \( \omega = 0.5 \), and \( \lambda = -0.3578 \). For these and all other results of the paper, calculations were
validated by checking conservation of mass to more than three significant figures.

A plot of $\lambda(\omega)$ for a range of $\omega$ values is given in Figure 6. For selected values of $\omega$, marked as hollow symbols on the curve of Figure 6, the alluvial channel responses are further documented in Figure 7. In Figure 7, similarity profiles are shown, normalized with respect to the evolving scaling variable $\xi(t)$ rather than with respect to a fixed length $L$. The figures illustrate the contrasted responses obtained depending on the value of parameter $\omega$. Values $\omega < 0$ correspond to underloading. The sediment supply $Q$ is below the equilibrium transport capacity $DS_2$ of the downstream alluvial channel, and degradation results. The alluvial edge is gradually washed downstream. At value $\omega = 0$, the upstream supply is precisely equal to the equilibrium capacity of the alluvial channel, and there is no geomorphic change; this scenario corresponds to the classical ‘graded river’ of Mackin (1948). Values $\omega > 0$ then correspond to overloading. The sediment supply is above the equilibrium transport capacity of the downstream alluvial channel, and deposition results at the transition. The alluvial edge moves upstream, gradually draping sediment over the bedrock floor. When $\omega > 1$, the sediment supply starts to exceed the equilibrium transport capacity $DS_1$ of the bedrock channel itself. Sediment is deposited before reaching the transition, covering the bedrock from upstream to downstream. Foreshadowing this complete change of behavior, the speed at which the bedrock-alluvial transition moves upstream becomes infinite as value $\omega = 1$ is approached from below.

The bedrock-alluvial case provides a good illustration of the limitations and power of the
analytical approach. The limitations clearly include the rather idealized assumptions which must be made at the onset: diffusive behavior, constant diffusion coefficient and straight initial profiles. The solution further applies only to spatial scales greater than the backwater length. In practice, deviations from normal depth are to be expected in the immediate vicinity of the slope break associated with the transition. On the photograph of Figure 2A, a water pool is observed between the bedrock reaches and the alluvial bed. This suggests that the uppermost tongue of the alluvial deposits displayed in Figure 5 will be scoured away to some extent by more realistic water flows. Provided that these limitations are accepted, however, the payoff is also quite substantial. First, exact explicit solutions are obtained, with calculations reduced to a simple root-finding step. Secondly, although the bedrock-alluvial transition is subject to both vertical aggradation/degradation and longitudinal migration, the approach succeeds in isolating a single dimensionless parameter $\omega$ controlling the overall response. This is a much greater reduction in complexity than could be achieved using dimensional arguments alone.

**LAKE BREACHING**

The second problem that can be solved using similarity is the lake breaching case illustrated in Figure 1B. The problem can also be seen as an idealization of the overtopping failure of a dam or levee. It involves water discharging from a lake into an alluvial channel, leading to incision at the lake outlet. Lake lamination effects are neglected: discharge contributions due to water level variations are assumed to be small compared to the lake inflow. The water discharge is thus assumed constant through the lake and its downstream alluvial channel. At the downstream rim of the lake, the local details of the critical section are neglected, and water is taken to switch from a nearly horizontal backwater profile to a normal flow profile along the sloping channel bed. Although this sharp transition between standing and running water may appear unrealistic, the landslide dam photograph of Figure 2B suggests that the
approximation is acceptable for shallow flow past a well-defined lake rim. Turning to the morphodynamic evolution, clear water is assumed to exit from the lake, incising the outlet channel and leading to the simultaneous lowering and upstream migration of the channel rim. The water level in the lake is assumed to ebb in lockstep with the decreasing elevation of its outlet. We note here another limitation of the theory: the adopted transport law does not include any threshold. In fact, actual bed material motion may require sufficiently high water discharges. For the example of Figure 2B, comparison with earlier photographs indicate that the lake outlet did not noticeably evolve over one month of low flow conditions. Higher discharges will be needed in order to dislodge the boulder assembly of the landslide dam and initiate lake breaching. Subject to these limitations, we now pursue the implications of the theory.

For similarity to apply, idealized initial conditions are again adopted, as illustrated in Figure 8. The submerged lake bed is given a constant adverse slope $S_1 < 0$, while the downstream alluvial channel is given an inclination $S_2 > 0$ and is assumed semi-infinite in extent. A sharp break in slope occurs at the lake rim, positioned initially at the origin $(x,z)=(0,0)$. The rim position $s(t)$ and its elevation then evolve in time due to the geomorphic action of the draining water. The above assumptions lead to the following mathematical problem:

$$\frac{\partial z}{\partial t} - D \frac{\partial^2 z}{\partial x^2} = 0, \quad s(t) < x < \infty, \quad (36)$$

$$z = -S_1 x, \quad \frac{\partial z}{\partial x} = 0, \quad x = s(t), \quad (37, 38)$$

$$\frac{\partial z}{\partial x} = -S_2, \quad x \rightarrow \infty. \quad (39)$$

$$z = -S_2 x, \quad x > 0, \quad t = 0. \quad (40)$$

Here boundary condition (38) states that the lake releases clear water only. No sediment is supplied to the alluvial channel aside from the material it erodes from its own bed.
It turns out that this problem is mathematically identical to the bedrock-alluvial problem treated in the previous section. The only difference is a change of sign of the upstream slope \( S_1 \): instead of a positive bedrock slope \( S_1 > 0 \), here an adverse lake bed slope \( S_1 < 0 \) is considered. Similarity again implies that the rim position migrates according to

\[
s(t) = \lambda \xi(t) = 2\lambda \sqrt{Dt} .
\]  

(41)

Since retrogressive erosion of the lake rim is expected, we anticipate a negative scaling constant \( \lambda < 0 \). The rate of retrogression is however unknown and must be found as part of the solution.

The derivation of the mathematical solution is in all respects identical to the bedrock alluvial case. The alluvial profile evolution can again be expressed in the similarity form

\[
z(x,t) = f(\sigma)\xi(t) ,
\]  

(42)

where \( \sigma = x/\xi(t) \) and the similarity function \( f(\sigma) \) is given by

\[
f(\sigma) = S_2 \left\{ -\sigma + \frac{[\exp(-\sigma^2) + \sqrt{\pi} \sigma \{\text{erf}(\sigma) - 1\}] } {\sqrt{\pi} \{\text{erf}(\sigma) - 1\}} \right\} .
\]  

(43)

The eigenvalue condition is identical to the one encountered earlier:

\[
\lambda - \omega \frac{\exp(-\sigma^2) + \sqrt{\pi} \sigma \{\text{erf}(\sigma) - 1\}} {\sqrt{\pi} \{\text{erf}(\sigma) - 1\}} = 0 ,
\]  

(44)

although here the parameter \( \omega \) reduces to \( \omega = S_2/(S_2 - S_1) \) and is restricted to the range \( 0 \leq \omega < 1 \). The corresponding values of scaling constant \( \lambda \) can be read from the same plot (Fig. 6) used previously for the bedrock-alluvial case, and it is clear that restriction \( \omega > 0 \) does indeed imply \( \lambda < 0 \).

The channel response described by this analytical solution is illustrated in Figure 9. The profiles shown are calculated assuming a ratio of upstream to downstream slope set to
$S_2/S_1 = -3$, which corresponds to parameter value $\omega = 1/4$, marked as a black diamond on the curve of Figure 6. The calculated value of the scaling constant is $\lambda = -0.1562$. Profiles are again plotted in dimensionless form, for equally spaced values of similarity variable $\xi(t)$. Viewed as a function of time, the morphodynamic evolution slows down as time progresses. More specifically, the migration rate of the lake rim $\frac{ds}{dt} = \frac{\lambda \sqrt{D}}{t}$ scales with $t^{-1/2}$.

Much like the bedrock-alluvial channel of Figure 5, the outlet channel depicted in Figure 9 undergoes degradation and is concave in shape. The two cases differ in that the transition migrates downstream in the bedrock-alluvial case, whereas it moves upstream in the lake breaching problem.

**PROGRADING DELTA**

The third problem considered is the prograding delta illustrated in Figure 1C. This is a semi-infinite variant of the prograding delta problem treated earlier by Voller et al. (2004) and shown in Figure 3B. As sketched in Figure 8, an alluvial channel of infinite length and constant slope $S_1 = S_2 = S$ is half-submerged at time $t = 0$ by a body of standing water of surface level $\zeta$. A delta forms, with a submerged foreset of slope $R$. Assumed constant, this slope corresponds to the angle of repose in the case of a Gilbert delta (Kostic and Parker 2003). To make the problem more general, we let the lake level evolve as a function of time. For similarity to be preserved, however, the time dependence cannot be arbitrary. Instead, it is restricted to be of the form

$$\zeta(t) = \mu \xi(t) = 2\mu \sqrt{Dt},$$

where $\mu$ is a dimensionless constant which controls the rate of lake level change. Negative values $\mu < 0$ correspond to a falling lake level, while positive values $\mu > 0$ correspond to a rising level. The special case of constant water level can of course be retrieved by setting
\( \mu = 0 \). At first sight, this special type of time dependence may seem artificially contrived. In the next section, however, it will be shown that it arises naturally in certain circumstances.

The corresponding mathematical problem is formulated as follows:

\[
\frac{\partial z}{\partial t} - D \frac{\partial^2 z}{\partial x^2} = 0, \quad -\infty < x < s(t), \\
\frac{\partial z}{\partial x} = -S, \quad x \to -\infty, \\
z = \mu \zeta(t), \quad x = s(t), \\
- D \frac{\partial z}{\partial x} = \frac{1}{2} \frac{d}{dt} \left\{ \frac{(Rs + z_2)(Ss + z)}{R - S} \right\}, \quad x = s(t), \\
z(x) = -Sx, \quad t = 0.
\]

where \( s(t) = \lambda \zeta(t) = 2\lambda \sqrt{D t} \). The most significant difference between this problem and the problems of the previous sections lies in the downstream boundary condition (49). This condition equates the sediment discharge supplied by the alluvial channel at the edge of the water body with the rate of change of the underwater sediment volume. It constitutes a sedimentary analogue of the Stefan condition used in melting and solidification problems, and can be derived from geometrical arguments (Swenson et al. 2000). Despite its apparent complexity, this condition satisfies the similarity requirements outlined earlier.

The solution of the problem proceeds along the same lines as before. Expressed in terms of ratio \( \sigma = x/\zeta(t) \) and similarity function \( f(\sigma) \) the problem reduces to:

\[
f''''(\sigma) + 2\sigma f''(\sigma) - f(\sigma) = 0, \quad -\infty < \sigma < \lambda, \\
f'(-\infty) = -S, \\
f(\lambda) = \mu, \quad f'(\lambda) = -2 \frac{(R\lambda + \mu)(S\lambda + \mu)}{R - S}. \quad (53, 54)
\]

This reduced problem has solution
\[ f(\sigma) = -A\sigma + B[\exp(-\sigma^2) + \sqrt{\pi}\sigma\text{erf}(\sigma) + 1], \]  \tag{55}

obtained by setting \( C = 1 \) in eq. (14). Coefficients \( A \) and \( B \) are obtained from boundary conditions (52) and (53), yielding

\[ A = S, \quad B = \frac{\mu + S\lambda}{\exp(-\lambda^2) + \sqrt{\pi}\lambda[1 + \text{erf}(\lambda)]}. \]  \tag{56}

Finally, Stefan condition (54) leads to the following transcendental equation for the scaling constant \( \lambda \):

\[ 2 \frac{\alpha}{\alpha - 1} \lambda(\lambda + \beta / \alpha) + \frac{\sqrt{\pi}\lambda[1 + \text{erf}(\lambda)]}{\exp(-\lambda^2) + \sqrt{\pi}\lambda[1 + \text{erf}(\lambda)]} - \frac{\lambda}{(\lambda + \beta)} = 0. \]  \tag{57}

The equation involves two independent control parameters \( \alpha = R / S \) and \( \beta = \mu / S \), which respectively normalize the foreset slope \( R \) and level evolution parameter \( \mu \) with respect to the initial channel slope \( S \).

Deferring to the next section the case of variable water level, results for a lake of constant level \( (\mu = 0) \) are plotted in Figure 11. The ratio of foreset slope to initial slope is set equal to \( \alpha = R / S = 5 \), yielding for the scaling constant the value \( \lambda = 0.4189 \). The solution illustrated in Figure 11 is similar to the solution of Voller et al. (2004) depicted earlier (Fig. 3B). The key difference is that here a semi-infinite channel is considered, hence the depositional topset extends inland upstream of the origin. In practical applications, the solution could be used to predict the deposit volumes emplaced in three different locations: 1) into the lake and under water; 2) into the lake but above the water level; 3) inside the alluvial channel upstream of the initial water edge. In field conditions, however, various complications can be expected to arise. First, even when the valley is narrow and constrained laterally by bedrock walls, three-dimensional patterns of water and sediment motion will influence the deposit shape. The photograph of Figure 2, for instance, shows an oblique delta front which will only be roughly approximated by the one-dimensional theory. Natural deltas
will further feature more complicated foresets and bottomsets, alimented by richer processes than the angle of repose avalanching assumed here (see e.g. Kostic and Parker 2003).

**SIMULTANEOUS LAKE BREACHING AND BACKFILL**

Beyond the elementary examples of the previous sections, the solution approach can be extended to composite problems featuring two or more elementary cases evolving over separate sub-domains. Such composite problems can be solved by the same methods as before as long as the similarity sub-domains have not yet started to encroach on each other. The example chosen to illustrate this approach is loosely inspired from the recent evolution of the Tsaolin landslide dam in Taiwan. This natural dam formed by the 1999 Chi-chi earthquake created a large temporary lake (Chen 1999; Li et al. 2002), which has since decayed under the geomorphic action of successive floods. The gradual disappearance of the lake involved a combination of two processes: 1) downstream, incision of the loose natural dam deposit by water flowing out of the lake; and 2) upstream, a gradual backfill from sediment carried into the lake by stream flow and debris flows.

An idealized version of this lake evolution problem can be tackled mathematically by considering two sub-domains connected by a lake of evolving water level. This is illustrated in Figure 12. The downstream sub-domain features a triangular landslide dam undergoing the breaching process treated earlier (see Fig. 9). Upstream of the lake, an alluvial channel delivers its sediment load to the lake, gradually building up a delta deposit. The water level $\zeta(t)$ in the lake connecting the two sub-domains is controlled by the evolution of the downstream dam crest. Because the downstream evolution is itself driven by a self-similar diffusive process, the resulting time dependence does satisfy the similarity requirement (45). As a result, the morphodynamic evolution of the upstream sub-domain reduces to the elementary problem treated in the previous section: a self-similar delta formation problem
with falling water level \((\mu < 0)\).

For the calculations shown in Figure 12, the dam is given the same triangular shape used previously to obtain the lake breaching profiles of Figure 9. The resulting lake level evolution is given by

\[ \zeta(t) = \mu \xi(t) = -\lambda S_1 \xi(t), \]  

where \(\lambda\) and \(S_1\) are respectively the scaling constant and adverse slope of the similarity solution for the dam evolution (see Fig. 9), both taking negative values. The identity \(\mu = -\lambda S_1\) derives from boundary condition (37) applied to the lake outlet. Lake lamination effects are again neglected, hence the evolution shown makes the implicit assumption that the extra water flux due to the gradual emptying of the lake is negligible compared to the constant water discharge flowing through the system.

Upstream of the lake, the prograding delta starts from the same initial conditions as the one shown earlier in Figure 11. Because the water level falls instead of staying constant, however, the delta in Figure 12 advances faster into the lake. The modified Stefan condition applied at the front further affects the overall shape of the upstream alluvial channel. As long as the toe of the delta does not reach the upstream face of the dam, the self-similarity of the sub-domain profiles is preserved. The approach obviously breaks down thereafter, and other techniques are needed to model the ensuing evolution. Nevertheless, the fact that it is possible to go so far using purely analytical methods constitutes a rather encouraging result, and suggests that yet other problems may be tackled by similar means.

CONCLUSIONS

In this paper, mathematical solutions were derived for several sedimentary problems involving a combination of alluvial channels, bedrock reaches and lakes. Before discussing
the advantages and possible uses of these solutions, the rather drastic assumptions needed to derive them must again be underscored. First, one-dimensional alluvial channels of semi-infinite extent evolving under diffusional sediment transport were assumed. This view is valid only if the profile evolution is examined over length scales that are short compared to the overall length of the channel, but long compared to the channel width and backwater length. It implies a sharp contrast between zones of standing water, where the surface is nearly horizontal, and zones of running water where the water depth quickly converges towards the normal depth. Other restrictive assumptions include simple initial profiles, a constant channel width, and steady supplies of water and sediment.

Provided that these limitations are accepted, however, the approach yields a number of benefits. First, exact solutions are obtained, free from the accuracy limitations of numerical calculations. Because the solutions come from a transparent analytical procedure instead of an algorithmic black box, they also provide a clearer view of the role of various boundary conditions and control parameters. Most importantly, the explicit solutions clarify the link between alluvial profile evolution and the migration of channel boundaries. They also highlight the common mathematical structure underlying geomorphic problems as diverse as bedrock-alluvial transitions, lake breaches and prograding deltas.

It is hoped that the proposed solutions will be helpful for various purposes. Comparison of their predictions with laboratory data could allow a better appraisal of the diffusional theory of fluvial morphodynamics. For quantitative comparisons, the idealized conditions of the theory can be reproduced in laboratory experiments using narrow flumes with parallel walls, as in the experiments of Muto (2001). Flumes of sufficient length will nevertheless be needed to approximate semi-infinite reaches. The solutions may also be used to validate numerical methods before applying the latter to more realistic problems. Alternatively, the solutions
themselves could be helpful for back-of-the-envelope analyses of field events, yielding typical time scales, modes of evolution, and distributions of deposit volumes once rough estimates of slopes and diffusion coefficients are available. Finally, the existence of these solutions in addition to the previously known results suggests that the family of explicit analytical solutions may be larger than expected, and that a continued search should be worthwhile.

ACKNOWLEDGEMENTS
Discussions and field trips kindly arranged by Prof. S.C. Chen of the National Chung-Hsing University, Taiwan, and Dr. C.H. Lai of the Water Resources Agency, Taiwan, provided inspiration for the present work and suggested in particular the Tsaolin example of the last section. Prof. T.C. Chen of the National Pingtung University, Taiwan, supplied information regarding the recent landslide dam across the Chihpen River. Special thanks are extended to Dr. J.B. Swenson of the University of Minnesota, Duluth, for his detailed comments and suggestions. Financial support from the National Science Council and the Council of Agriculture, Taiwan, is also gratefully acknowledged.
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### Table 1. List of symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Dimensions*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A, B, C$</td>
<td>Constants of integration</td>
<td></td>
</tr>
<tr>
<td>$\alpha, \beta$</td>
<td>Parameters controlling the motion of the delta front</td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>Ordinary differential</td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td>Diffusion coefficient</td>
<td>$L^2T^{-1}$</td>
</tr>
<tr>
<td>exp</td>
<td>Exponential function</td>
<td></td>
</tr>
<tr>
<td>erf</td>
<td>Error function</td>
<td></td>
</tr>
<tr>
<td>$f(\sigma)$</td>
<td>Shape function of the self-similar profile</td>
<td></td>
</tr>
<tr>
<td>$f'(\sigma), f''(\sigma)$</td>
<td>First and second derivatives of the shape function</td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>Unspecified function of a single argument</td>
<td></td>
</tr>
<tr>
<td>$g, h$</td>
<td>Functions of reduced coordinate $\sigma$</td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>Index of general invariant solution</td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td>Position of channel boundary</td>
<td>$L$</td>
</tr>
<tr>
<td>$Q$</td>
<td>Volumetric sediment supply (per unit width)</td>
<td>$L^2T^{-1}$</td>
</tr>
<tr>
<td>$R$</td>
<td>Slope of the delta foreset</td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>Initial channel slope</td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
<td>$T$</td>
</tr>
<tr>
<td>$u$</td>
<td>Arbitrary diffused scalar</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>Horizontal coordinate</td>
<td>$L$</td>
</tr>
<tr>
<td>$y$</td>
<td>Variable of integration</td>
<td></td>
</tr>
<tr>
<td>$z = z(x,t)$</td>
<td>Channel bed elevation above a reference datum</td>
<td>$L$</td>
</tr>
<tr>
<td>$\lambda = s/\xi$</td>
<td>Reduced horizontal coordinate of moving boundary</td>
<td></td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>Lake level evolution parameter</td>
<td></td>
</tr>
<tr>
<td>$\sigma = x/\xi$</td>
<td>Reduced horizontal coordinate</td>
<td></td>
</tr>
<tr>
<td>$\xi = \xi(t)$</td>
<td>Evolving length scale of the self-similar profile</td>
<td></td>
</tr>
<tr>
<td>$\omega$</td>
<td>Sediment supply parameter</td>
<td></td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Lake level elevation above a reference datum</td>
<td></td>
</tr>
<tr>
<td>$\partial$</td>
<td>Partial differential</td>
<td></td>
</tr>
</tbody>
</table>

*The notations for the dimensions are $L =$ length, $T =$ time, blank = dimensionless.
**FIGURE CAPTIONS**

Fig. 1. Idealized problems involving alluvial channels with a moving boundary on one end and the other boundary at infinity: **A**) bedrock-alluvial transition; **B**) lake breaching; **C**) delta formation. In each case, \( s(t) \) denotes the position of the moving boundary.

Fig. 2. Field examples of alluvial channels with moving boundaries: **A**) exposed bedrock (photo foreground) upstream of an alluvial reach (starting at the left bend in the background): Tahan River upstream of Shihmen Dam, Northern Taiwan, May 2003 (photo by Wei Chia Yu); **B**) abrupt transition between standing and running water at the crest of a landslide dam: Chihpen River, Eastern Taiwan, November 2005 (photo by Yueh-Jen Lai); **C**) delta front prograding into standing water: Ronghua Dam, Northern Taiwan, November 2005 (photo by A.T.H. Perng). The dashed line in the background shows the location of the delta front in August 2004.

Fig. 3. Two known similarity solutions for evolving shorelines: **A**) the classical Neumann solution (see e.g. Crank 1984), cast as a delta problem; **B**) the recent delta solution of Voller, Swenson and Paola (2004). Both solutions feature a fixed boundary condition at the origin and a moving boundary at the shoreline. Dashed lines show the initial lake bed; thin lines are snapshots of the alluvial profile for equally spaced values of the square root of the elapsed time \( \sqrt{t} \).
Fig. 4. Definition sketch for bedrock-alluvial transition problem: slope $S_1$ of the bedrock floor, initial slope $S_2$ of the downstream alluvial reach, and upstream sediment influx $Q$.

Fig. 5. Profile evolution for bedrock-alluvial transition: A) underloading case with zero upstream sediment supply ($S_2/S_1 = 0.5$, $\omega = -1$); B) overloading case ($S_2/S_1 = 0.2$, $\omega = 0.5$). Long dashes: bedrock basement; short dashes: initial profile of the downstream alluvial channel; continuous lines: successive snapshots of the alluvial channel profile for $\xi(t)/L = 1, 2, 3, 4, 5$, where $\xi(t) = 2\sqrt{Dt}$.

Fig. 6. Bedrock-alluvial transition. Dependence of the scaling constant $\lambda = s(t)/(2\sqrt{Dt})$ on supply parameter $\omega = (Q/D - S_2)/(S_1 - S_2)$. Profiles are shown on Fig. 7 for the conditions denoted by symbols $\triangle$, $\square$, $\bigcirc$, $\triangledown$. Values $\omega < 0$ imply depletion and values $\omega > 0$ imply accretion upstream of the alluvial channel. The same curve can be used for the lake breaching problem, and symbol $\blacksquare$ corresponds to the conditions of Fig. 9.

Fig. 7. Bedrock-alluvial transition. Positions of the moving boundary for various values of supply parameter $\omega = (Q/D - S_2)/(S_1 - S_2)$ are denoted by the following symbols: $(\triangle) \omega = -1; (\square) \omega = -1/2; (\bigcirc) \omega = 0; (\triangledown) \omega = 1/2$. Continuous lines are the corresponding alluvial channel profiles, and the dashed line is the underlying bedrock.

Fig. 8. Definition sketch for lake breaching problem: the upstream face of the dam has
adverse slope $S_1 < 0$ and the downstream lake outlet channel has initial slope $S_2$.

Fig. 9. Profile evolution for lake breaching process ($\omega = 1/4$). Short dashes: initial profile of the erodible dam; continuous lines: successive snapshots of the alluvial channel profile for $\xi(t)/L = 1, 2, 3, 4, 5$, where $\xi(t) = 2\sqrt{Dt}$. Erosion of the dam crest leads to lake drainage.

Fig. 10. Initial conditions for the prograding delta problem. The water level $\zeta$ in the downstream lake starts from zero but may evolve according to time history $\zeta(t)$.

Fig. 11. Profile evolution for semi-infinite alluvial channel reaching a body of standing water ($\alpha = R/S = 5$) of constant level ($\mu = 0$). Short dashes: initial channel profile; continuous lines: successive snapshots of the alluvial profile for $\xi(t)/L = 1, 2, 3, 4, 5$, where $\xi(t) = 2\sqrt{Dt}$. A delta builds up, gradually displacing the shoreline.

Fig. 12. Profile evolution for the simultaneous upstream backfill and downstream breaching of a natural lake, obtained by combining similarity solutions for breaching (see Fig. 9) and delta formation in a lake of falling water level. Short dashes: initial bed profile; continuous lines: successive snapshots of the alluvial profile for $\xi(t)/L = 1, 2, 3, 4, 5$, where $\xi(t) = 2\sqrt{Dt}$. 

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FIGURES

Figure 1
Figure 3

Figure 4
Figure 7

Figure 8

Figure 9
Figure 10

Figure 11

Figure 12