Groundwater drainage and recharge by networks of irregular channels

Wei-Jay Ni and Hervé Capart

Department of Civil Engineering and Hydrotech Research Institute,
National Taiwan University, Taiwan

Abstract

A linear complementarity theory is proposed for the coupled treatment of groundwater seepage and surface runoff along a sloping plane ground perturbed by irregular channels. Steady downslope routing is applied to the two-dimensional overland flow, while Green functions are used to relate the three-dimensional groundwater motion to the surface drainage and recharge distribution. The coupling between the surface and subsurface components is formulated as a linear complementarity problem. The algorithms needed to compute useful solutions are presented and validated. Two variants of the theory are addressed, one applicable to large scale problems in which capillary effects can be neglected, the other to small scale processes for which the range of variation of the topography does not exceed the capillary fringe. Linearized theories for these two cases are found to be very close in their structure and in their predictions, providing a rigorous basis for the representation of field scale processes by laboratory scale analogues. Predictions from the theory are compared with detailed experiments, in which groundwater sapping was monitored using a video camera and a laser sheet scanning apparatus.

AGU Index Term: 1830 (Groundwater/surface water interaction).
1. Introduction

Surface channels formed along hill slopes, river plains, and beach faces typically derive their water supply from direct precipitation, upstream tributaries, or tidal pool drainage. In certain cases, however, the key contribution to runoff comes instead from groundwater seepage. Compared to other runoff contributions, seepage is special in that it can take the dual role of source and sink. Groundwater may seep out from shallow aquifers and feed networks of irregular channels. Conversely, surface water may seep back into the ground, possibly depleting streams until they run dry. Examples from the field can be used to introduce the issues involved. The photographs of Figures 1 and 2 depict river beds in Taiwan where interactions between groundwater and surface water can be observed.

Groundwater seepage may affect river beds in a number of ways. After a flood, water stored within the valley alluvium may gradually drain out, as observed in the small braiding plain of Fig. 1. In another scenario (see Fig. 2), a low stage river may function in a hybrid regime, with a base flow that repeatedly infiltrates and exfiltrates along the watercourse. This may restrict the surface flow to a set of disconnected channels, with water supplied upstream by groundwater resurgence disappearing back downstream into the alluvial fill. Figure 2 illustrates the significant flow rates that can bypass through the subsurface before reappearing as surface runoff. There is no surface link between the main upstream channel and the two oblique secondary channels. Yet the latter convey substantial flow rates which become apparent as they drop past the overfall of a check dam.

The groundwater-fed surface runoff may or may not exert a geomorphic action on the bed. It may actively incise its own network of channels, through groundwater sapping at the point of resurgence or fluvial erosion further downstream. Alternatively, the groundwater-fed runoff may be passively channelized into pre-existing bed depressions formed during high stage
flows (see Fig. 2). The example of Fig. 1 is of mixed type, in which a fluvially shaped braiding plain is being reworked by the water flow draining out from the ground.

Both in the field and in the laboratory, water seepage contributions to streamflow have been extensively studied by hydrologists. Early studies of stream-aquifer links include works by Boussinesq [1877, 1904] and by Tóth [1963]. The role of ground saturation in the generation of runoff was clarified by the pioneering catchment studies of Dunne and Black [1970]. More recently, water exchanges through permeable channel beds and margins were studied in the field by McCarthy et al. [1992] and Harvey and Bencala [1993]. At larger scales, the seasonal expansion and contraction of groundwater-fed stream networks was examined by de Vries [1995]. The phenomena have also been probed in laboratory sandbox and field plot studies [e.g., Abdul and Gillham, 1984; VanderKwaak and Loague, 2001]. Winter [1995] and Sophocleous [2002] provide comprehensive reviews of past and current research devoted to interactions between groundwater and surface water.

For almost a century, surface channelization due to the erosive action of groundwater seepage has also been of interest to geomorphologists. Studies seeking to clarify the basic mechanisms include, from the earliest to the most recent, Jaggar [1908], Wentworth [1928], Dunne [1990], Schörghofer et al. [2004], and Lobkovsky et al. [2004]. In the field, gully formation by groundwater sapping has been documented on beaches by Otvos [1999], and Schörghofer et al. [2004]. Large-scale analogues of these small-scale valleys have been suggested by some researches on the Colorado Plateau [Laity and Malin, 1985], in Hawaii [Kochel and Piper, 1986], in Florida [Schumm et al., 1995] and in Japan [Onda, 1994]. Groundwater sapping has even been hypothesized to play a geomorphic role on Mars [Higgins, 1982; Malin and Edgett, 2000; Aharonson et al., 2002]. Laboratory experiments have been used in a number of studies [Jaggar, 1908; Kochel et al., 1985; Howard and McLane, 1988; Howard, 1988; Gomez and
More extensive introductions to past and current research on the geomorphic role of groundwater seepage are provided by Brown and Bradley [1995] and Sidle and Honda [2004].

In the various circumstances described above, the interaction of groundwater and surface flow is a key focus of interest. A number of modeling tools have consequently been developed for their joint treatment. The more computationally intensive approaches use shallow water equations to describe the surface flow, coupled with a groundwater component treated in three dimensions by the Richards equation [e.g., VanderKwaak and Loague, 2001; Morita and Yen, 2002], or reduced to two horizontal dimensions using the vertically averaged Dupuit equation [e.g., Anderson, 2005; Gunduz and Aral, 2005]. More analytically inspired approaches include the linear complementarity description of Aitchison et al. [1983], the conceptual model of de Vries [1995], and the conformal mapping treatment of Anderson [2003]. The latter studies are especially close to the point of view adopted in the present work.

In the above studies and most of the published literature, however, a number of restrictive assumptions are typically made. The scope is generally limited to two-dimensional slices of ground, or to three-dimensional geometries having a smooth ground surface. It is often further assumed that the water seeping out from the ground is immediately removed, with no opportunity to re-infiltrate the aquifer. In field cases, these assumptions may be highly unrealistic. As illustrated on Figures 1 and 2, groundwater seepage may feed intricate channel networks incised into irregular ground, and both influent and effluent seepage can play important roles. In the present work, we seek to address these challenges using a simple, analytically-based treatment.

The paper is structured as follows. Section 2 describes the governing equations and boundary
conditions adopted to describe the coupled groundwater–overland flow system. The linear complementarity theory derived from these equations is presented in sections 3 and 4. Two variants of the theory are considered. The capillarity-free variant described in section 3 applies to unconfined aquifers under the assumption that capillarity plays no role. The capillarity-confined variant described in section 4, on the other hand, applies when the range of variation of the surface topography never exceeds the capillary rise. The computational algorithms needed to construct solutions are presented in section 5. Section 6 documents the laboratory sandbox experiments used to test the theory. In section 7, the theoretical predictions are compared with the laboratory observations. Finally, some conclusions and avenues for further work are proposed.
2. Governing equations and boundary conditions

As sketched in the introduction and illustrated on Fig. 3, the present work focuses on problems involving both overland and groundwater flow. Instead of blanketing the entire ground surface, the overland runoff is restricted to limited zones of surface flow which tend to coalesce into a network of channels. This surface network is activated by water seeping out from the aquifer, and may be depleted further downstream by water seeping back into the ground. The two-way seepage leads to groundwater drainage and recharge. It also controls the spatial extent of the channel network, free to shrink or expand depending on the water supply. To describe this problem mathematically, governing equations must be provided for the overland and groundwater components, and supplemented by boundary conditions. It will be assumed throughout that these boundary conditions evolve sufficiently slowly for the problem to be considered steady at any given time.

As shown on Fig. 3, we adopt a sloping coordinate system \((x, y, z)\), tilted at angle \(\beta\) corresponding to the inclination of a reference plane. Measured with respect to this reference plane, the topography of the ground surface is denoted by \(z(x, y)\). The corresponding height with respect to a horizontal datum is

\[
h^{(s)}(x, y) = -x \sin \beta + z(x, y) \cos \beta .
\]  

Consider first the overland flow component. Assuming steady state, the surface water discharge \(q^{(w)}(x, y) = (q_x^{(w)}, q_y^{(w)})\) must satisfy the continuity equation

\[
\nabla_{xy} \cdot q^{(w)} = \frac{\partial q_x^{(w)}}{\partial x} + \frac{\partial q_y^{(w)}}{\partial y} = s ,
\]  

where the seepage rate \(s(x, y)\) represents the groundwater contribution to the overland flow. Since both effluent and influent seepage are considered, this term may be a source \((s > 0)\) or a sink \((s < 0)\).
We assume that the ground surface features no local minima in elevation, hence ponding does not occur. The depth of the sheet of flowing water is neglected compared to topography variations. Under these assumptions, surface water can be routed in a simple way by postulating that the flow systematically proceeds downhill along the steepest available path, i.e. that

$$ q^{(w)} = q^{(w)} j^{(s)}, $$

(3)

where the flow direction $ j^{(s)}(x,y) $ depends only on the local topography according to

$$ j^{(s)} = - \frac{\nabla_{xy} h^{(s)}}{\|\nabla_{xy} h^{(s)}\|}. $$

(4)

Here and above $ \nabla_{xy} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) $ is the two-dimensional gradient operator, applicable to the surface vector fields $ q^{(w)}(x,y) $ and $ j^{(s)}(x,y) $. The scalar flow rate $ q^{(w)}(x,y) $ is further subject to the important requirement

$$ q^{(w)}(x,y) \geq 0. $$

(5)

The surface channels can run dry ($ q^{(w)} = 0 $ is allowed), but surface runoff cannot flow in the upslope direction ($ q^{(w)} < 0 $ is forbidden).

Turning to the groundwater component, the water motion within the aquifer is governed by Darcy’s law:

$$ \mathbf{U} = -K \nabla \Phi, $$

(6)

where $ \mathbf{U} = (U_x, U_y, U_z) $ is the specific discharge, $ K $ is the permeability (assumed homogeneous and isotropic), $ \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) $ is the three-dimensional gradient operator, and

$$ \Phi(x,y,z) = \frac{P}{\gamma} + h $$

is the hydraulic head. Here $ \gamma $ is the specific weight of water and

$$ h = -x \sin \beta + z \cos \beta $$

again represents the height above a reference datum. Under the
assumption of constant porosity, the continuity equation is
\[ \nabla \cdot \mathbf{U} = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} = 0. \] (7)

Upon substitution of Darcy’s equation, one obtains the Laplace equation
\[ \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \] (8)
implying that groundwater flow is a potential flow. Boundary conditions remain to be formulated. The domain is assumed to be infinite in the longitudinal and transverse directions \( x \) and \( y \). At its base, the aquifer may either be infinite in depth or bounded by an impervious floor at finite depth \( z = -D \) below the reference plane.

The conditions near the ground surface, at the top of the aquifer, require special attention. The water table, or phreatic surface, is defined following Bear [1972] as the surface \( z^{(w)}(x, y) \) on which the pressure \( p \) is atmospheric. In zones of overland flow, we neglect the thickness of the water layer flowing along the surface, and it is again assumed that no ponding occurs. Thus the water table either lies below or coincides with the ground surface, leading to a second inequality
\[ z^{(w)} \leq z^{(s)}. \] (9)
Saturation excess overland flow is assumed [Dunne and Black, 1970; Dunne and Leopold 1978]: surface flow is activated only where the water table reaches the ground level. The two inequalities (5) and (9) are therefore related by the complementarity condition
\[ (z^{(s)} - z^{(w)}) q^{(w)} = 0. \] (10)
The condition holds that, at any point along the ground surface, at least one of the two inequalities must reduce to an equality. Overland flow is turned off \( (q^{(w)} = 0) \) wherever the water table is below ground \( (z^{(w)} < z^{(s)}) \). Conversely, the two surfaces coincide and equality \( z^{(w)} = z^{(s)} \) must hold wherever the overland flow is active \( (q^{(w)} > 0) \). Groundwater seepage is likewise restricted to the zones of overland flow where \( z^{(w)} = z^{(s)} \). In these zones,
groundwater recharge is described as a surface source term \( r(x, y, z = z^{(w)}) = r(x, y) \), positive for recharge and negative for drainage. Where the water table meets the ground surface, continuity requires that

\[ r = -s. \]  

In other words term \( s(x, y) \) viewed as a source from the point of view of the overland flow becomes a sink from the point of view of the aquifer. Influent seepage occurs where \( r > 0 \) \((s < 0)\), and conversely effluent seepage occurs where \( r < 0 \) \((s > 0)\).

In zones where no overland flow occurs, a soil layer of variable saturation will exist between the water table and the ground surface [Gillham, 1984]. A detailed description of the water movement in that layer would require a partially saturated treatment, using for instance Richards’ equation [see e.g. VanderKwaak and Loague, 2001; Morita and Yen, 2002]. An alternative, approximate approach [Bear 1972; Lobkovsky et al. 2004] is to introduce a tension-saturated capillary fringe of finite height \( h_c \) above the phreatic surface. Within this fringe, the pore pressure becomes negative but the flow continues to be described by the Laplace equation (8). In the present work, we will idealize the situation even further by considering only the two limiting cases schematized on Fig. 4.

The first limiting case, illustrated on Fig. 4a, is a capillarity-free aquifer. In this limit, the capillarity height \( h_c \) is neglected compared to the topography variations of interest,

\[ h_c \ll \Delta z^{(s)}. \]  

The soil above the water table is then considered dry (or, more realistically, but equivalently when the flow is steady, any water retained in the soil above the water table is considered motionless). Continuity applied to the water table leads to boundary condition:

\[ -\mathbf{U} \cdot \mathbf{n}^{(w)} = r, \quad \text{on} \quad z = z^{(w)}(x, y), \]
where \( \mathbf{n}^{(w)} = (-\partial z^{(w)}/\partial x, -\partial z^{(w)}/\partial y, 1) \) is the upward normal to the water table and \( r(x, y) \) is the rate of recharge introduced previously. Upon substitution of the normal vector, the kinematic boundary condition applied along the water table becomes

\[
U_x \frac{\partial z^{(w)}}{\partial x} + U_y \frac{\partial z^{(w)}}{\partial y} - U_z = r, \quad \text{on} \quad z = z^{(w)}(x, y).
\] (14)

In zones where no recharge or drainage occurs \( (r = 0) \), the phreatic surface constitutes a material interface across which no water flux takes place. This limit is appropriate for coarse ground and large scale problems, such as those illustrated on the photos of Figures 1 and 2. However it is not appropriate for small-scale field processes or for the laboratory experiments described below in section 6.

At small scales, another limiting case occurs when the depth of the water table below the ground surface is everywhere smaller than the capillary height

\[
z^{(x)} - z^{(w)} < h_c.
\] (15)

In that case the Laplace equation (8) applies to the entire ground domain, both above and below the water table. Water is free to move across the phreatic surface. In zones of no overland flow \( (r = 0) \), it is the ground surface \( z = z^{(x)} \) which becomes a no flux boundary where \( \mathbf{U} \cdot \mathbf{n}^{(x)} = 0 \). Here \( \mathbf{n}^{(x)} = (-\partial z^{(x)}/\partial x, -\partial z^{(x)}/\partial y, 1) \) is the upward normal to the ground surface. To account for drainage and recharge, the kinematic boundary condition applied along the ground becomes

\[
U_x \frac{\partial z^{(x)}}{\partial x} + U_y \frac{\partial z^{(x)}}{\partial y} - U_z = r, \quad \text{on} \quad z = z^{(x)}(x, y).
\] (16)

Water can only percolate out of the ground where the pore pressure is atmospheric. In zones of negative water pressure \( (z^{(w)} < z^{(x)}) \), the ground becomes a capillary exposed surface [see Bear, 1972] along which the water is retained in the soil due to capillary suction. The term “capillarity-confined aquifer” will be used to refer to this second limiting case, shown on Fig. 4b. This case is of interest in particular for laboratory work. In small-scale sandbox...
experiments, it is unrealistic to impose condition (12), but the soil and flow parameters can be chosen to meet condition (15) throughout the domain. The top boundary condition thus constitutes an important difference between the two cases. For the capillarity-free aquifer, the kinematic boundary condition is applied along the water table, the elevation of which is unknown and must be found as part of the solution. By contrast for the capillarity-confined aquifer it is applied along the ground surface, which is assumed known. From this point of view, only the capillarity-free case constitutes a genuine free surface problem.

There is another sense, however, in which both cases constitute free boundary problems: along the ground surface, the zones of overland flow have unknown extent. Where overland flow occurs, the water table is known (it coincides with the ground surface), but the recharge intensity is unknown. The opposite occurs in zones of no overland flow, where the recharge intensity is zero, but the elevation of the phreatic surface is unknown. Regardless of the role exerted by capillarity, the top surface of the problem therefore involves two distinct zones, along which different types of conditions apply. The limit between the two zones is an unknown of the problem and constitutes itself a free boundary. To complicate matters further, there is no guarantee that this boundary will adopt simple shapes, or even that the boundary curve will be continuous: disconnected, arbitrarily shaped zones of overland flow appear to be the norm rather than the exception in practical cases.

To tackle this free boundary challenge, our approach will be to cast the problem in a revised formulation. This will involve linearizing the groundwater equations, then representing the aquifer response using fundamental solutions of the linearized equations, and finally coupling this response with the overland flow distribution. Because the surface boundary condition affects this process at various junctures, the capillarity-free and capillarity-confined cases will be treated separately in the next two sections.
3. Linearized theory for capillarity-free aquifer

We consider perturbations with respect to a simple base state: parallel groundwater motion in the downslope direction. The water table of the base flow has inclination $\beta$ and coincides with the reference plane $z = 0$. The overall potential is then decomposed into base flow and perturbed potentials:

$$\Phi(x, y, z) = -x \sin \beta + \phi(x, y, z),$$

and the specific discharge components become

$$U_x = K \sin \beta + u_x, \quad U_y = u_y, \quad U_z = u_z,$$

where

$$ (u_x, u_y, u_z) = -K \nabla \phi.$$  \hfill (19)

The base flow can be checked to satisfy the governing equations exactly in the absence of drainage and recharge. Accordingly, the perturbed component represents deviations from the base flow induced by distributed drainage and recharge along the top surface of the aquifer.

For the capillarity-free case examined in this section, an infinite aquifer depth $D$ is assumed. Boundary conditions are thus needed along the water table only. Using the above decomposition into base flow and perturbation, the system of equations governing the groundwater flow can be written:

Laplace: \quad $\nabla^2 \phi = 0,$ \quad $z < z^{(w)}(x, y),$ \hfill (20)

Kinematic BC: \quad $(K \sin \beta + u_z) \frac{\partial z^{(w)}}{\partial x} + u_y \frac{\partial z^{(w)}}{\partial y} - u_z = r,$ \quad $z = z^{(w)}(x, y),$ \hfill (21)

Dynamic BC: \quad $\frac{p}{\gamma} = \phi - z \cos \beta = 0,$ \quad $z = z^{(w)}(x, y).$ \hfill (22)

Using (19) and (22), the free surface kinematic boundary condition (21) can be rewritten in terms of potential $\phi$ alone:

$$\left( K \sin \beta - K \frac{\partial \phi}{\partial x} \right) \frac{\partial \phi}{\partial x} - K \left( \frac{\partial \phi}{\partial y} \right)^2 + K \cos \beta \frac{\partial \phi}{\partial z} = r \cos \beta,$$ \quad on \quad $z = z^{(w)}(x, y).$  \hfill (23)
The above equation highlights the difficulty of the full non-linear problem: not only does the kinematic boundary condition involve squared partial derivatives, it must also be applied along a free surface of unknown elevation.

To proceed, the equations are simplified by assuming that the perturbations around the base flow are small. To make this assumption precise, equations are formulated in terms of dimensionless variables

\[ x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{L}, \quad z^{(w)}' = \frac{z^{(w)}}{A}, \quad \phi' = \frac{\phi}{A}, \quad r' = \frac{r}{R}, \]  

(24)

where \( L \) is a length scale of the problem, \( A \) denotes a typical amplitude of the water table perturbations, and \( R = AK / L \) is the corresponding representative recharge rate. For completeness, the ground surface and overland flow rate are also made non-dimensional using

\[ z^{(s)}' = \frac{z^{(s)}}{A} \quad \text{and} \quad q^{(w)}_s = \frac{q^{(w)}}{RL} = \frac{q^{(w)}}{AK}. \]  

(25)

For the groundwater component, the Laplace equation does not change, while the boundary conditions (21) and (22) become

\[ \sin \beta \frac{\partial \phi'}{\partial x'} + \cos \beta \frac{\partial \phi'}{\partial z'} - A \frac{L}{R} \left( \left( \frac{\partial \phi'}{\partial x'} \right)^2 + \left( \frac{\partial \phi'}{\partial y'} \right)^2 \right) = r' \cos \beta, \]  

(26)

\[ \phi' = z^{(w)}' \cos \beta, \]  

(27)

to be applied along

\[ z' = \frac{A}{L} z^{(w)}. \]  

(28)

All dimensional parameters drop out from the equations to the exception of ratio

\[ \varepsilon = \frac{A}{L} = \frac{R}{K} \]  

(29)

which denotes a typical slope of the water table perturbations or, equivalently, a dimensionless recharge rate. If one assumes that this parameter is small \( \varepsilon << 1 \), then two crucial simplifications ensue. First, the squared partial differential terms drop out of the kinematic boundary condition (26). Secondly, both the kinematic and dynamic boundary
conditions can be applied along reference plane \( z' = 0 \) instead of the unknown free surface. It will however be necessary to keep in mind that these approximations hold only for mild sloped perturbations and gentle recharge or drainage.

Dropping the primes for convenience, the set of dimensionless linearized equations governing the groundwater motion can be written \cite{Polubarinova-Kochina, Dagan}

\begin{align*}
\text{Laplace:} & \quad \nabla^2 \phi = 0, \quad z < 0, \quad (30) \\
\text{Kinematic free surface BC:} & \quad \tan \beta \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} = r, \quad z = 0, \quad (31) \\
\text{Dynamic free surface BC:} & \quad \phi = z^{(w)} \cos \beta, \quad z = 0. \quad (32)
\end{align*}

Having linearized the equations, the principle of superposition becomes applicable. This can be exploited to construct an integral representation of the flow. Using the sifting property of the delta function, the recharge distribution \( r(x, y) \) can be represented by the convolution

\[
r(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(\xi, \eta) \delta(x - \xi, y - \eta) d\xi d\eta,
\]

and the corresponding aquifer response can be expressed as

\[
\phi(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(\xi, \eta) \hat{\phi}(x - \xi, y - \eta, z) d\xi d\eta,
\]

\[
z^{(w)}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(\xi, \eta) \hat{z}^{(w)}(x - \xi, y - \eta) d\xi d\eta,
\]

where \( \hat{\phi}(x, y, z) \) and \( \hat{z}^{(w)}(x, y) \) are the fundamental solutions, or Green functions of the problem. These functions represent the steady aquifer response to a point recharge at the origin. They are obtained by solving problem (30)-(32) after replacing the recharge \( r(x, y) \) by the delta function \( \delta(x, y) \). The steady fundamental solution for the potential is

\[
\hat{\phi}(x, y, z) = \frac{1}{2\pi} \frac{\tan \beta (x - z \tan \beta)(x^2 + y^2 + z^2)^{1/2} + (x^2 + y^2 - xz \tan \beta)(1 + \tan^2 \beta)^{1/2}}{(x - z \tan \beta)^2 + (1 + \tan^2 \beta)y^2} \frac{(1 + \tan^2 \beta)^{1/2}}{(x^2 + y^2 + z^2)^{1/2}}. \quad (36)
\]

It is derived by time integration of an unsteady Green function originally derived by Dagan [1966] and extended to sloping aquifers using a transformation proposed also by Dagan in
another work [Dagan, 1967a]. A detailed derivation based in part on symmetry methods is provided in Appendix. The corresponding fundamental solution for the water table is

\[ z^{(w)}(x, y) = \frac{1}{2\pi} \frac{x \tan \beta (x^2 + y^2)^{1/2} + (x^2 + y^2)(1 + \tan^2 \beta)^{1/2}}{\cos \beta (x^2 + (1 + \tan^2 \beta)y^2)(1 + \tan^2 \beta)^{1/2}(x^2 + y^2)^{1/2}}. \]  

(37)

If the water table of the unperturbed aquifer is horizontal \( (\beta = 0) \), one retrieves the results

\[ \hat{\phi}(x, y, z) = \frac{1}{2\pi} \frac{1}{(x^2 + y^2 + z^2)^{1/2}}, \quad z^{(w)}(x, y) = \frac{1}{2\pi} \frac{1}{(x^2 + y^2)^{1/2}}. \]  

(38)

given by Dagan [1967a,b], and the potential coincides with the classical Neumann’s function for the Laplace equation in the lower half-space [see e.g. Kevorkian, 1990]. For a sloping aquifer, the three-dimensional linearized response to a point recharge of unit strength described by Green functions (36) and (37) is illustrated on Fig. 5. The response is shown superposed onto the base flow, for an aquifer of slope \( \beta = 14 \) degrees corresponding to the experiments of section 6 below.

The integral representation (35) for the water table level can now be used to couple the groundwater and overland flow components. Using interface condition (11), the overland flow continuity equation (36) can be written

\[ r = -s = -\nabla_{xy} \cdot \mathbf{q}^{(w)}. \]  

(39)

Substitution of this expression into the integral representation (35) then yields

\[ z^{(w)}(x, y) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\nabla_{\xi\eta} \cdot \mathbf{q}^{(w)}) \hat{z}^{(w)}(x - \xi, y - \eta) d\xi d\eta \]  

(40)

where \( \nabla_{\xi\eta} = (\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}) \) is used to evaluate the overland flow divergence at the point of recharge \( (\xi, \eta) \). Using integration by parts, this equation can be rewritten

\[ z^{(w)}(x, y) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nabla_{\xi\eta} \cdot \mathbf{q}^{(w)}(\xi, \eta) \hat{z}^{(w)}(x - \xi, y - \eta) d\xi d\eta \]

\[ + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{q}^{(w)}(\xi, \eta) \cdot \nabla_{\xi\eta} \hat{z}^{(w)}(x - \xi, y - \eta) d\xi d\eta. \]  

(41)

Invoking the divergence theorem, the first integral can be transformed into a contour integral.
along the domain boundary. Since the original integral is over the whole plane, the domain boundary is sent to infinity. We then assume that the far away state of the aquifer does not influence the local groundwater conditions, hence the contribution from this contour integral vanishes. In the second integral, the downslope flow relation (3) can be used to factorize the overland discharge vector in the form \( q^{(w)} = q^{(w)} j^{(s)} \), where \( j^{(s)} \) is the downslope direction vector set by the ground topography, and \( q^{(w)} \geq 0 \) is the scalar overland flow rate. Finally, inequality (9) and complementarity condition (10) are needed to complete the reformulation of the problem.

The end result is the following linear complementarity formulation of the coupled groundwater/overland flow problem (for a capillarity-free aquifer):

\[
\begin{align*}
    z^{(w)}(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q^{(w)}(\xi,\eta) j^{(s)}(\xi,\eta) \cdot \nabla_{\xi,\eta} z^{(w)}(x-\xi, y-\eta) \, d\xi \, d\eta \\
    z^{(w)} &\leq z^{(s)}, \quad q^{(w)} \geq 0, \quad (z^{(s)} - z^{(w)})q^{(w)} = 0.
\end{align*}
\] (42) (43)

Under the assumptions laid out above, this surprisingly compact set of equations yields a complete description of the coupled groundwater/overland flow system, and constitutes the main theoretical result of the present work. The problem involves two unknown scalar fields \( z^{(w)}(x,y) \) and \( q^{(w)}(x,y) \), i.e. the water table level and the overland flow rate. These two fields are related to each other by linear integral relation (42), and further constrained by complementary inequalities (43). A known ground topography \( z^{(s)}(x,y) \) must of course be provided to fully specify the problem.

These different elements of the formulation can now be discussed in some more detail. First, the linear integral relation (42) describes the water table response to distributed overland flow along the ground surface. This response is expressed in terms of doublets (or dipole)
fundamental solutions \( \partial \bar{z}(w) / \partial \xi \) and \( \partial \bar{z}(w) / \partial \eta \) obtained by differentiating the fundamental solution (37) for the response to a point recharge. The corresponding groundwater flow fields are illustrated on Fig. 6. Note that because the base flow and recharge interact through free surface boundary condition (31), the fundamental solutions for the source and doublets are distorted in the downstream direction. This implies in particular that the doublets \( \partial \bar{z}(w) / \partial \xi \) and \( \partial \bar{z}(w) / \partial \eta \) are not simply rotated aliases of each other, as would be the case for zero base flow.

The next element of the formulation is the pair of complementary inequalities (43). These separate inequalities must be satisfied by the unknown fields over the entire domain, with the additional constraint that at least one of the two inequalities must reduce to an equality at any given point. In zones of overland flow, equality \( z'(w) = z'(x) \) is activated, whereas equality \( q'(w) = 0 \) applies to zones of no overland flow. As a consequence of the free boundary challenge outlined earlier, the choice of which equality applies at a given point must be determined as part of the solution. Finally, the ground topography (assumed known) can be seen to intervene at two junctures: first, it sets the direction of the downslope overland flow through vector \( j'(x) = -\nabla_{xy} h'(x) / \| \nabla_{xy} h'(x) \| \); secondly, it controls the water table elevation at which seepage sets in through inequality \( z'(w) \leq z'(x) \) and complementarity condition \( (z'(x) - z'(w)) q'(w) = 0 \).

Problems of this kind, known collectively as linear complementarity problems [Elliott and Ockendon, 1982] are encountered in a variety of applications. These include plastic deformation, electropainting, and elastic contact problems (where they are known as Signorini problems after Signorini [1933]). A linear complementarity formulation of groundwater percolation has been proposed earlier by Aitchison et al. [1983]. The formulation proposed
here differs from this work in several respects. First, Aitchison and co-workers use a partial differential formulation rather than an integral formulation, and deal with a 2D rather than a 3D problem. Most importantly, Aitchison et al. [1983] consider percolation outflow only, and assume that water which has percolated out of the aquifer vanishes from the system. In our notations, they impose constraint \( s(x, y) \geq 0 \) rather than \( q^{(w)}(x, y) \geq 0 \). The field photos of the introduction and laboratory experiments of section 6 show that this mathematical assumption is often not appropriate physically. The groundwater which seeps out can also percolate back in after flowing along the surface. The theory proposed above takes this observation into account.

Assuming that the above linear complementarity problem can be solved, the information that is obtained includes the water table elevation \( z^{(w)}(x, y) \) and the overland flow distribution \( q^{(w)}(x, y) \). One can also retrieve the seepage distribution \( s(x, y) = -r(x, y) \) from (39), as well as the groundwater motion throughout the aquifer using integral relation (36) for the three-dimensional potential \( \phi(x, y, z) \). For idealized ground topographies, it is not impossible that special solutions of this problem could be obtained by analytical methods. In the present paper, however, our focus is on networks of irregular channels, and solutions will be sought instead by computational means. The algorithms required for this purpose will be presented in section 5. Before that, the next section is devoted to the treatment of the capillarity-confined case.
4. Linearized theory for capillarity-confined aquifer

For the capillarity-confined case, we consider an aquifer of finite depth, with an impervious floor at depth \( z = -D \) below the reference plane. The groundwater motion is again assumed to involve perturbations about a downslope base flow. As before, the water table of this base flow has inclination \( \beta \) and coincides with the reference plane \( z = 0 \). Now however perturbations are induced by ground surface irregularities \( z^{(s)}(x,y) \) in addition to the distributed recharge \( r(x,y) \) along the surface. The equations governing the perturbed potential \( \phi(x,y,z) \) become (in dimensional variables):

\[
\text{Laplace: } \nabla^2 \phi = 0, \quad -D < z < z^{(s)}, \quad (44)
\]

\[
\text{Bottom kinematic BC: } u_z = 0, \quad z = -D, \quad (45)
\]

\[
\text{Top kinematic BC: } \left\{ \begin{array}{l}
\sin \beta - \frac{\partial \phi}{\partial x} \frac{\partial z^{(s)}}{\partial x} - \frac{\partial \phi}{\partial y} \frac{\partial z^{(s)}}{\partial y} + \frac{\partial \phi}{\partial z} = \frac{r}{K}, \\
\quad z = z^{(w)}.
\end{array} \right. \quad (46)
\]

Because the flow is confined between known top and bottom boundaries, conditions are no longer applied along an unknown free surface. The nature of the condition to be applied along the top boundary (ground surface), however, depends on the relative position of the water table. The phreatic level \( z^{(w)}(x,y) \) is expressed implicitly as the locus

\[
z^{(w)}(x,y) \cos \beta = \phi(x,y,z^{(w)}(x,y)), \quad (47)
\]

where the pore pressure is atmospheric ( \( p = 0 \)). Along the ground surface, recall from section 2 that atmospheric pressure is assumed ( \( p^{(s)} = 0 \) or equivalently \( \phi(x,y,z^{(s)}) = z^{(s)} \cos \beta \)) where seepage occurs ( \( r \neq 0 \)). A no flux condition ( \( r = 0 \)) applies instead if the water table is below ground ( \( z^{(w)} < z^{(s)} \)). Although the problem is not a classical free surface problem, the condition along the top boundary may thus switch from Dirichlet (prescribed potential) to Neuman (prescribed flux) depending on the local phreatic level. To deal with this difficulty, it is again advantageous to linearize the equations.

Using the dimensionless variables introduced previously in definitions (24) and (25), the top
kinematic BC becomes
\[ \sin \beta \frac{\partial z^{(s)}}{\partial x} + \frac{\partial \phi'}{\partial z'} - \frac{A}{L} \left\{ \frac{\partial \phi'}{\partial x'} \frac{\partial z^{(s)}}{\partial x'} + \frac{\partial \phi'}{\partial y'} \frac{\partial z^{(s)}}{\partial y'} \right\} = r', \]  
(48)
and is to be applied along \( z' = \frac{A}{L} z^{(s)} \). A truncated Taylor expansion can also be used to approximate condition (47) in the form
\[ z^{(w)}(x', y') \cos \beta = \phi'(x', y', z^{(w)}) \approx \phi'(x', y', 0) + \frac{A \, \partial \phi'}{L \, \partial z'}(x', y', 0) z^{(w)}'. \]  
(49)
If ratio \( \varepsilon = A / L \) is again assumed to be small, then three valuable simplifications arise. First, the partial differential products can be neglected in condition (48). Secondly, this condition can be applied along the reference plane instead of the perturbed ground surface. Finally, the second term in expansion (49) (and higher order terms) can be neglected and a simple relation is obtained between the water table elevation and the surface potential.

Dropping the primes for convenience, the dimensionless linearized equations governing the groundwater motion in the capillarity-confined case are:

Laplace: \[ \nabla^2 \phi = 0, \]
\[ -d < z < 0, \]  
(50)
Bottom kinematic BC: \[ \frac{\partial \phi}{\partial z} = 0, \]
\[ z = -d, \]  
(51)
Top kinematic BC: \[ \frac{\partial \phi}{\partial z} = r - \sin \beta \frac{\partial z^{(s)}}{\partial x}, \]
\[ z = 0, \]  
(52)
where \( d = D / L \). Furthermore, the phreatic level is given by
\[ z^{(w)}(x, y) = \frac{\phi(x, y, 0)}{\cos \beta}. \]  
(53)
Because it represents a known forcing due to ground surface irregularities, term \( \sin \beta \frac{\partial z^{(s)}}{\partial x} \) has been moved to the right hand side of condition (52). In the linearized problem, local rises and drops in the ground surface are equivalent to sink and source terms, respectively. The potential problem above is standard, and its solution can be represented by the convolution.
\[ \phi(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ r(\xi, \eta) - \sin \beta \frac{\partial \phi^{(s)}}{\partial \xi} \right\} \phi(x - \xi, y - \eta, z) d\xi d\eta \]  

where the fundamental solution \( \hat{\phi} \) is the response of an aquifer confined between two parallel planes \(-d < z < 0\) to a point recharge at the origin \( r(x, y) = \delta(x, y) \). For a confined aquifer of infinite depth, the fundamental solution would again be the point source potential \( \hat{\phi}(x, y, z) = \frac{1}{2\pi} \frac{1}{(x^2 + y^2 + z^2)^{1/2}} \).  

For an aquifer of finite dimensionless depth \( d \), the fundamental solution is obtained by the method of images as the response to a periodic array of point sources spaced at intervals \( 2d \) along the \( z \)-axis:  

\[ \hat{\phi}(x, y, z) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \frac{1}{\left\{ x^2 + y^2 + (z - 2kd)^2 \right\}^{1/2}}. \]  

This guarantees that (except for the singularity at the origin) the no flux condition is identically satisfied along the top and bottom planes. An obvious difficulty with expression (56) is that the infinite sum is divergent. This reflects the fact that there is no valid solution to the finite depth problem for an isolated source. The difficulty disappears, however, provided that the flow results from a neutral distribution of sources and sinks. The simplest case involves a point source and point sink of equal and opposite strengths, separated by a finite distance. The corresponding groundwater flow pattern is illustrated on Fig. 7. Doublet flow results upon letting the distance between source and sink vanish while increasing the strength. Neutrality will be insured below by expressing the excitation exclusively in terms of such doublets or source/sink pairs. When using fundamental solution (56), the context of a neutral distribution will always be assumed.

Using relation (53), the solution for the phreatic surface is given by  

\[ z^{(w)}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ r(\xi, \eta) - \sin \beta \frac{\partial \phi^{(s)}}{\partial \xi} \right\} z^{(w)}(x - \xi, y - \eta) d\xi d\eta, \]
where the fundamental solution for the water table is \( \hat{z}^{(w)}(x,y) = \hat{\phi}(x,y,0)/\cos\beta \). We stress that the fundamental solutions are not the same for the capillarity-free and capillarity-confined cases. When needed, subscripts “cf” and “cc” and the notations \( \hat{\phi}_{cf}, \hat{z}^{(w)}_{cf} \) and \( \hat{\phi}_{cc}, \hat{z}^{(w)}_{cc} \) will be used to distinguish between the two.

Integral representation (57) can now be used to couple the groundwater and surface flow components, using the same approach that was used earlier for the capillarity-free case. It is convenient to first rewrite the integral in the form

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\eta d\xi z^{(w)}(x-x, y-y, \eta, \xi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(\xi, \eta) \hat{z}^{(w)}(x-x, y-y, \eta, \xi) d\xi d\eta
\]

where

\[
\hat{z}^{(w)}(x,y) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin\beta \frac{\partial z^{(w)}}{\partial \xi} z^{(w)}(x-x, y-y, \eta, \xi) d\xi d\eta
\]

is the response of the phreatic surface to the irregularities of the ground topography in the absence of any drainage or recharge \( (r = 0) \). Replacing the recharge distribution by the divergence of the overland flow, then integrating by parts, we obtain the doublet integral

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nabla \cdot q^{(w)}(\xi, \eta) \cdot \nabla \hat{\phi}_{cc} z^{(w)}(x-x, y-y, \eta, \xi) d\xi d\eta,
\]

linking the phreatic surface to the distribution of overland flow. To close the problem, factorization \( q^{(w)} = q^{(w)} j^{(s)} \), inequalities (5) and (9), and complementarity condition (10) are recalled. The result is a modified linear complementarity formulation of the coupled groundwater / overland flow problem, applicable to the capillarity-confined case:

\[
\hat{z}^{(w)}(x,y) = \hat{z}^{(w)}_{0}(x,y) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q^{(w)}(\xi, \eta) j^{(s)}(\xi, \eta) \cdot \nabla \hat{\phi}_{cc} z^{(w)}_{cc}(x-x, y-y, \eta, \xi) d\xi d\eta,
\]

\[
z^{(w)} \leq z^{(s)}, \quad q^{(w)} \geq 0, \quad (z^{(s)} - z^{(w)}) q^{(w)} = 0.
\]

The linearized formulation for the capillarity-confined aquifer is thus found to share essentially the same structure as the capillarity-free formulation examined earlier. The two unknown fields are again the phreatic surface \( z^{(w)}(x,y) \) and overland flow rate \( q^{(w)}(x,y) \),
subject to the same constraints as before. Modifications intervene only in the integral relation (61) linking the phreatic surface to the overland flow. Two modifications have been made to this relation. First, the fundamental solution is now the capillarity-confined version $z_{cc}^{(w)}$. Beyond this simple swap, the influence of the base flow on the perturbations has changed. Whereas for the capillarity-free case the base flow interacted with the water table $z^{(w)}$ and affected the fundamental solution $z_{cf}^{(w)}$, here the base flow interacts instead with the ground surface $z^{(s)}$ leading to an independent term $z_{0}^{(w)}$. From a computational point of view, these changes are relatively minor. In the next section, the algorithms needed to solve the two linear complementarity problems will therefore be presented jointly.
5. Computational scheme

Up to this point, a continuum point of view has been adopted to describe the overland and groundwater flows. For computational purposes, a discrete description will now be introduced. The overland flow will be described using an 8-neighbor scheme, commonly used to route runoff over digital terrain models (DTMs). The groundwater motion, on the other hand, will be treated by the method of fundamental solution (MFS). Once discretized, the coupled formulation will take the form of an algebraic linear complementarity problem (LCP). The special iteration method devised to solve this algebraic problem will finally be presented.

For simplicity, all surface fields are sampled on a Cartesian grid

\[(x_i, y_j) = (j\Delta x, i\Delta y), \quad i = 1, \ldots, I, \quad j = 1, \ldots, J\]  \hspace{1cm} (63)

where \(\Delta x = \Delta y\). Ground elevations \(z_{yx}^{(s)} = z^{(s)}(x_j, y_i)\) and phreatic levels \(z_{yx}^{(w)} = z^{(w)}(x_j, y_i)\) are assigned to the nodes of the grid. If desired, they can be stored in two-dimensional arrays. To save on indices, however, it is convenient to use a renumbering scheme and label each grid node using a single index

\[k = (i-1)I + j, \quad k = 1, \ldots, K \quad \text{where} \quad K = IJ.\] \hspace{1cm} (64)

For instance, \(z_{k}^{(s)} = z^{(s)}(x_k, y_k)\) will be used hereafter instead of \(z_{yx}^{(s)}\) to refer to the ground level at a given node of the two-dimensional grid.

On this Cartesian grid, the overland flow component is treated using an 8-neighbor scheme \([O \text{ Callaghan and Mark}, 1984; Jenson and Domingue, 1988]\). The grid nodes are connected to some of their North, Northeast, East, Southeast, South, Southwest, West, and Northwest neighbors by linear channel segments. Each node \(i\) can receive inflows from more than one upstream neighbor \(j\), but the full outflow will be routed to a single downstream neighbor \(j^*(i)\). The latter is chosen by selecting the path of maximum downwards slope \(-\nabla h^{(s)}:\)
\[ j^*(i) = \text{argmax}_j \frac{h_i^{(s)} - h_j^{(s)}}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} \]  

(65)

where it is recalled that \( h^{(s)} = -x \sin \beta + z^{(s)} \cos \beta \) in sloping coordinates, and where the operation \( \text{argmax} \) yields the argument \( j \) for which the slope assumes its maximum. Variable \( Q_i \) will be used to denote the outflow discharge from node \( i \) delivered to downstream node \( j^*(i) \). By construction, these outflow discharges are subject to the constraint

\[ Q_i \geq 0. \]  

(66)

As illustrated on Fig. 8, this approach amounts to replacing the downslope vector field \( \mathbf{j}^{(s)}(x,y) \) by a discrete channel network. The overland continuity equation (2) can then be translated into an algebraic equation for each node

\[ Q_i - \sum_j C_{ij} Q_j = S_i, \]  

(67)

where \( S_i = s(x_i, y_i) \Delta x \Delta y \) is a discrete source term at node \( i \), representing effluent seepage through a small ground patch of area \( \Delta x \Delta y \). The coefficients \( C_{ij} \) are elements of a connectivity matrix \( \mathbf{C} \), set equal to 1 if node \( j \) routes its discharge to node \( i \), and equal to zero otherwise. The discrete continuity equation is simple to interpret: the outflow discharge from a given node is equal to the sum of inflow discharges plus the seepage rate from the subsurface.

Before treating the more complicated coupled case in which both the \( Q_j \) and \( S_i \) are unknown, it is instructive to see how one can solve for the discharges \( Q_j \) when the sources \( S_i \) are given. This case would be encountered if the unknown seepage distribution was replaced by a known rainfall distribution routed as direct runoff. A brute force approach would be to assemble linear system

\[ \mathbf{J} \mathbf{Q} = \mathbf{S}, \]  

(68)

obtained by stacking as many equations of the form (67) as there are nodes in the grid, then
solve by Gaussian elimination. Due to the special structure of the problem, a much more economical approach can be adopted. Matrix $J$ has diagonal elements $J_{ii}$ equal to 1, and off-diagonal elements $J_{ij} = -C_{ij}$ equal either to $-1$ or 0, depending on whether or not node $j$ discharges to node $i$. Furthermore, it can be transformed into a lower triangular matrix simply by reordering the nodes from highest to lowest elevation $h^{(i)}$. This is because by construction, the outflow from a given node is routed to another node of lower elevation. By adopting this reordering, the problem can thus be solved using a single back-substitution sweep. In practice, it is not even necessary to assemble matrix $J$. One simply proceeds from the highest node to the lower node, accumulating discharge along the way using statement

$$Q_i = S_i + \sum_j C_{ij}Q_j.$$  \hspace{1cm} (69)

Because it is highly efficient, this method is widely used to route direct precipitation runoff over digital terrain models. We will present below a modified version devised specifically to treat the coupled overland/groundwater flow problem.

A computational treatment of the groundwater flow component must first be worked out. The objective is to discretize the integral representations derived in sections 3 and 4. In pursuing this objective, the main difficulty stems from the singular character of the fundamental solutions $\hat{Z}_{cf}^{(w)}$ and $\hat{Z}_{cc}^{(w)}$ at the source point. This difficulty can be addressed in two ways. In the boundary integral element method (BIEM) [Liggett and Liu, 1983; Gardner, 2004], surface sources are distributed along the boundary and integrated panel by panel. The singularity is then carefully integrated away when evaluating the influence of a panel on itself. In the method of fundamental solution (MFS) [Fairweather and Karageorghis, 1998; Young et al., 2006], by contrast, point-like sources are used, but they are placed outside the domain some distance away from the boundary. Boundary conditions can then be enforced at isolated collocation points placed on the boundary itself. An attractive feature of the MFS is that both
sources and boundary nodes can be treated as point-like entities. No connected mesh is required either inside the domain or along its boundary, and there is no need for a complicated integration scheme. Because this relative simplicity facilitates coupling with the other components of the problem, the MFS is adopted in the present work. The method of fundamental solution was earlier used to solve Signorini problems by Poullikkas et al. [1998a,b].

Applied to the present problem, the method of fundamental solution amounts to replacing integrals of the form

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(\xi, \eta) \phi(x - \xi, y - \eta, z) \, d\xi \, d\eta \]  

(70)

by the discrete sums

\[ \sum_{k} R_k \Phi_k(x - x_k, y - y_k, z) \]  

(72)

\[ \sum_{k} R_k \zeta^{(w)}_k(x - x_k, y - y_k) \]  

(73)

where index \( k = 1, \cdots, K \) runs over all nodes of the grid. In the expressions above, functions \( \Phi(x, y, z) \) and \( \zeta^{(w)}(x, y) \) are regularized versions of the fundamental solutions \( \phi(x, y, z) \) and \( \zeta^{(w)}(x, y) \). They represent contributions from discrete sources, centered at grid nodes \((x_k, y_k)\) and weighted by coefficients \( R_k = r(x_k, y_k) \Delta x \Delta y \) representing local recharge rates through a surface patch of area \( \Delta x \Delta y \). The regularized functions are obtained by introducing a small elevation offset \( \ell \) in the original fundamental solutions:

\[ \Phi(x, y, z) = \Phi(x, y, z - \ell) , \]  

(74)

\[ \zeta^{(w)}(x, y) = \frac{\Phi(x, y, 0)}{\cos \beta} = \frac{\Phi(x, y, -\ell)}{\cos \beta} . \]  

(75)

Thanks to the regularization length \( \ell \neq 0 \), the functions become well-behaved at the origin.
By construction, they continue to satisfy the Laplace equation inside the domain, and preserve
the relationship between water table and surface potential. In the above expressions, one must
of course use either the capillarity-free or capillarity-confined versions of the fundamental
solutions depending on the case of interest. Figure 9 illustrates the difference between the
original solutions $\hat{\phi}_{cf}$, $\hat{\phi}_{cc}$ and their regularized versions $\tilde{\phi}_{cf}$, $\tilde{\phi}_{cc}$. To satisfy the neutrality
condition, required for convergence in the case of a finite depth aquifer, a source-sink pair is
shown for the capillarity-confined case.

To illustrate and validate the approach, the Method of Fundamental Solutions is first applied
to a benchmark problem treated analytically by Dagan [1967a] and used by Liggett and Liu
[1983] to validate their non-linear BIEM computations. The problem involves a horizontal
aquifer of infinite depth subject to uniform recharge over a disk. The length scale $L$ and
recharge scale $R$ for this problem can obviously be chosen as the disk radius and recharge
intensity. In dimensionless form, the recharge distribution is then given by

$$ r(x, y) = \begin{cases} 1 & \text{if } \rho < 1, \\ 0 & \text{otherwise}, \end{cases} $$ (76)

where $\rho = \sqrt{x^2 + y^2}$ is the radial coordinate. The analytical solution for the water table is
given by [Dagan, 1967a]

$$ z^{(w)}(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 (\rho^2 + \sigma^2 - 2\rho\sigma\cos\varphi)^{1/2} \sigma \, d\sigma \, d\varphi. $$ (77)

It is obtained by integration over the unit disk of the fundamental solution (38) for an
infinitely deep ($D \to \infty$) horizontal aquifer ($\beta = 0$). The above recharge distribution and
analytical water table are illustrated on Fig. 10a,b. Note that for this special case, the solutions
for the capillarity-free and capillarity-confined cases are identical. Also, since the aquifer has
infinite depth, the source distribution does not need to be neutral. The method of fundamental
solution presented above is tested for two different problems, forward and inverse.
In the forward problem, the recharge \( r(x, y) \) is assumed known and the water table \( z^{(w)}(x, y) \) is computed. This is done simply by replacing the exact integral (77) by the discrete sum

\[
z_i^{(w)} = \sum_j R_j \tilde{z}^{(w)}(x_i - x_j, y_i - y_j).
\]  

(78)

where

\[
R_j = \begin{cases} 
\Delta x \Delta y & \text{if } \sqrt{x_j^2 + y_j^2} < 1, \\
0 & \text{otherwise},
\end{cases}
\]  

(79)

and

\[
\tilde{z}^{(w)}(x, y) = \frac{1}{2\pi} \frac{1}{(x^2 + y^2 + \ell^2)^{1/2}}.
\]  

(80)

It is found that best results are obtained when the regularization length is set to \( \ell = \frac{1}{2} \Delta x \), and this is the choice adopted for all the computations presented hereafter. Computational results for the forward problem are compared with the analytical solution on Fig. 10c. Even for a relatively coarse discretization \( \Delta x = \Delta y = 0.1 \), accurate results are obtained. Errors at positions \((x, y) = (0,0), (1,0), \) and \((2,0)\) are respectively 0.4, 4.7 and 0.1 % of the peak value \( z^{(w)}(0,0) = 1 \).

In the inverse problem, it is the water table \( z^{(w)}(x, y) \) which is assumed known, and the objective is to reconstruct the recharge distribution \( r(x, y) \). Using the MFS representation, this is done by solving the linear system

\[
\sum_j R_j \tilde{z}^{(w)}(x_i - x_j, y_i - y_j) = z_i^{(w)}
\]  

(81)

for the recharge strengths \( R_j \). The system can be cast in matrix form

\[
N R = Z^{(w)},
\]  

(82)

where vector \( Z^{(w)} \) has known components \( z_i^{(w)} \), vector \( R \) has unknown components \( R_j \), and matrix \( N \) has known coefficients \( N_{ij} = \tilde{z}^{(w)}(x_i - x_j, y_i - y_j) \). Matrix \( N \) is a full matrix that would be very expensive to invert by elimination methods. For this reason, we use
instead under-relaxed Jacobi iterations, which are particularly simple to implement. Starting from an initial guess or from values of \( R_j \) calculated at the previous iteration, each iteration involves the following two steps:

**Jacobi step:**

\[
R'_j = \frac{1}{N_i} \left\{ z^{(w)}_i - \sum_{j \neq i} N_j R_j \right\}, \tag{83}
\]

**Under-relaxation step:**

\[
R''_j = R'_j + \omega(R'_j - R_j), \tag{84}
\]

where \( \omega < 1 \) is an under-relaxation factor. For the uniform recharge inverse problem, reasonable convergence is obtained by setting \( \omega = 0.005 \), with initial values \( R_j \) set to zero over the entire domain. For the actual implementation, it is useful to rewrite (83) in the vector form

\[
\mathbf{R} = \{\mathbf{Z}^{(w)} - \sum_{j \neq i} R_j N_j^{(j)}\} \mathbf{N}^{-1}
\]

where \( N_j^{(j)} \) is the \( j \)-th column vector of matrix \( \mathbf{N} \), with the diagonal elements set equal to zero, \( \mathbf{N} \) is a vector composed of the diagonal elements of \( \mathbf{N} \), and operator \( ./ \) represents array division (element-by-element division of two vector arrays). Using this statement, the computer calculations can be fully vectorized, leading to substantial speed-up. Furthermore, the sum loop involves only the non-zero weights \( R_j \neq 0 \), leading to large savings when recharge sources are sparsely distributed. Finally, it can be noted that the vectors \( N_j^{(j)} \) do not need to be calculated or stored separately for each node \( j \), since they are based on translation-invariant fundamental solutions. Instead the fundamental solutions can be calculated only once on an extended grid of size \( (2I-1) \times (2J-1) \), from which vectors of size \( K \times K \) can be read off. It is thus never necessary to actually assemble matrix \( \mathbf{N} \), which would require allocating memory for an array of size \( K \times K \). For a moderately sized grid of 100×100 nodes, the resulting matrix would contain a prohibitive 10,000×10,000 elements.
Using the approach above, substantial savings in computing time and memory are achieved, and a standard PC was sufficient to carry out all the computations reported in the present paper. On Fig. 10d, inverse problem results are compared with the analytical recharge distribution for the case of uniform recharge. The grid used is the same grid of 100 × 100 nodes (resolution Δx = Δy = 0.1) used previously for the forward computations. As could be expected, results for the inverse problem are slightly less accurate than for the forward problem. In particular, an overshoot of +2.6% and an undershoot of −2.2% are observed near the perimeter ρ = 1 of the uniform recharge. Overall, however, the MFS performs well both for the forward and inverse calculations. This is important for the present work because the linear complementarity problem treated below involves a combination of forward and inverse problems. In zones of no overland flow, the water table is unknown (forward problem), whereas the recharge is unknown at nodes where overland flow occurs (inverse problem).

The overland and groundwater flow components can now be coupled together. For the capillarity-free case, the starting point is the linear complementarity formulation (42)-(43), which we now treat in detail. The treatment of the capillarity-confined case involves the same steps, which will not be repeated for the sake of conciseness. As a preliminary step, the doublet integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(\xi, \eta) \cdot \nabla \hat{z}^{(\omega)}(x - \xi, y - \eta) d\xi d\eta$$

is discretized using the regularized fundamental solutions introduced above. This is done by placing a sink and a source at the upstream and downstream ends of each discrete overland channel linking two grid nodes. The sink and source have equal and opposite strengths, set equal to the overland discharge \( Q_k \) linking node \( k \) and its downstream node \( j^*(k) \). This yields for the groundwater level the representation
\[ z^{(w)}(x, y) = \sum_k Q_k \Delta z^{(w)}_k(x, y), \quad (87) \]

where

\[ \Delta z^{(w)}_k(x, y) = z^{(w)}(x - x_{j(k)}, y - y_{j(k)}) - z^{(w)}(x - x_k, y - y_k). \quad (88) \]

In this way, the overland discharges \( Q_k \) become the primary unknowns of the problem. If needed, the seepage and recharge strengths can be retrieved from the discharges using

\[ S_k = -R_k = Q_k - \sum_j C_{kj} Q_j, \quad (89) \]

Using renamed indices, it follows that the water table elevation at node \( i \) can be expressed in terms of the outflow discharges at nodes \( j \) using the discrete sum

\[ z^{(w)}_i = \sum_j Q_j \Delta z^{(w)}_j(x_i, y_i), \quad (90) \]

in which index \( j \) runs over all nodes of the grid. This can be cast in the matrix form

\[ Z^{(w)} = MQ \quad (91) \]

where vector \( Z^{(w)} \) has components \( z^{(w)}_i \), vector \( Q \) has components \( Q_j \), and matrix \( M \) has coefficients \( M_{ij} = \Delta z^{(w)}_j(x_i, y_i) \). The overall result of the discretization procedure is to reduce formulation (42)-(43) to the following algebraic linear complementarity problem (LCP):

\[ \sum_j M_{ij} Q_j = z^{(w)}_i \leq z^{(s)}_i, \quad Q_j \geq 0, \quad (z^{(w)}_i - z^{(s)}_i)Q_i = 0. \quad (92) \]

Linear complementarity problems constitute constrained versions of the classical linear algebra problems \( AX = B \) [Elliott and Ockendon, 1982]. The solution involves partitioning vectors \( Q \) and \( Z^{(w)} \) into two complementary subsets. At nodes of the first subset (in zones of no overland flow), the \( Q_i \) are set equal to zero and the water table is calculated from the discharges \( Q_j \) at all the other nodes. At nodes of the second subset (zones of overland flow), by contrast, the water table \( z^{(w)}_i \) is constrained to equal the ground level \( z^{(s)}_i \), and the discharge \( Q_i \) must be chosen to satisfy a set of linear equations. Solving the LCP involves determining (if it exists) the partition and values of primary and secondary unknowns \( Q_i \) and \( z^{(w)}_i \) which simultaneously satisfy the complementary conditions. Once the problem is solved,
the active overland flow domain is determined as the union of all ground patches where the \(Q_i\) are non-zero.

The computational algorithms needed to solve this problem in practice must now be presented. The basic approach follows Elliott and Ockendon [1982], and involves iterative solutions of linear system \(\mathbf{MQ} = \mathbf{Z}^{(w)}\), interleaved with projection steps in which negative \(Q_i\) values are reset to zero. For the linear system iterations, under-relaxed Jacobi iterations using the simple steps (83)-(84) were initially tried [see Ni, 2005], but were found to converge only for test cases involving sparse overland flow activity. For cases involving denser networks of overland channels, the approach fails to converge. The reason why the Jacobi approach succeeds for inverse problem \(\mathbf{NR} = \mathbf{Z}^{(w)}\), but sometimes fails for problem \(\mathbf{MQ} = \mathbf{Z}^{(w)}\) can be traced to difference between matrices \(\mathbf{N}\) and \(\mathbf{M}\). Matrix \(\mathbf{N}\) is constructed from regularized fundamental solutions \(\tilde{z}^{(w)}\), which present a single peak at the source point. This leads to coefficients of maximum strength along the diagonal only. By contrast matrix \(\mathbf{M}\) is constructed from discrete dipole fundamental solutions \(\Delta\tilde{z}^{(w)}\), which feature a pit at node \(i\) and a peak at downstream node \(j^*(i)\). Coefficients of maximum strength are thus found both on and off the diagonal, defeating the diagonal-based Jacobi iterations. By construction, however, these pits and peaks are arrayed inside matrix \(\mathbf{M}\) according to the same pattern as the coefficients of overland flow matrix \(\mathbf{J}\) encountered earlier.

This diagnostic readily leads to the following revised approach, aiming to exploit the special structure of matrix \(\mathbf{M}\). We first express system \(\mathbf{MQ} = \mathbf{Z}^{(w)}\) as the column sum

\[
\mathbf{Z}^{(w)} = \sum_k Q_k \mathbf{M}^{(k)},
\]

where column vector \(\mathbf{M}^{(k)}\) represents the influence map associated with discharge \(Q_k\). We then decompose \(\mathbf{M}^{(k)}\) into two parts.
where \( I^{(k)} \) is the \( k \)-th column vector of the identity matrix \( I \), \( J^{(k)} \) is the \( k \)-th column vector of the overland flow matrix \( J \) introduced earlier in equation (68), and \( \tilde{M}^{(k)} \) represents a modified influence map obtained by setting the pit and peak values to zero. The linear system \( \tilde{M}Q = Z^{(w)} \) is then rewritten as

\[
\tilde{J}P = T = \sum_j Q_j \tilde{M}^{(j)} - Z^{(w)},
\]

where vector \( P \) has components \( P_j = M_j Q_j \). In this way both the pit and peak components of matrix \( \tilde{M} \) have been retained on the left-hand-side, with the weaker components transferred to the right-hand-side. Although matrix \( \tilde{J} \) is not diagonal, it was seen earlier that a system having structure \( \tilde{J}P = T \) can be solved in a single sweep upon reordering unknowns \( P_j \) from the highest to the lowest node. Based on this decomposition, the algorithm proposed to solve LCP problem (92) involves repeated iterations of the following steps:

- **accumulation step:**
  \[
  T = \sum_{j} Q_j \tilde{M}^{(j)} - Z^{(w)},
  \]
- **back-substitution step:**
  \[
  Q'_i = \{ T_i + \sum_j C_{ij} M_j Q_j \} / M_{ii},
  \]
- **projection step:**
  \[
  Q''_i = \max \{ Q'_i, 0 \},
  \]
- **under-relaxation step:**
  \[
  Q'''_i = Q + \omega (Q''_i - Q).
  \]

In the above loop, the vectorized accumulation step (96), back-substitution step (97) and under-relaxation step (99) are modified versions of statements (85), (69) and (84) encountered previously. Inserted within the back-substitution sweep, the projection step is meant to enforce positivity constraint \( Q_j \geq 0 \) during the iteration process [Elliott and Ockendon, 1982]. Convergence is monitored by calculating the residual

\[
\max_j \{ \sum_i Q_i \tilde{M}^{(j)} - z_i^{(s)} \}
\]

until it is driven below a specified threshold value. The revised algorithm presented above was applied to a variety of test cases stemming from the experimental data documented below.
In all test cases, satisfactory convergence was observed, regardless of the overland channel density. Like the simple Jacobi approach, the algorithm does not require the assembly of any large matrix, and runs on a standard PC for the grids used below, featuring approximately $100 \times 100$ nodes. Computed results will be presented and compared with experimental data in section 7. Before that, the sandbox experiments used to generate the test data are presented in the next section.
6. Laboratory experiments

The small-scale experiments which motivated the above theory can now be introduced. Carried out in the summer of 2003, the experiments involve an inclined sand layer which gradually evolves a channelized surface topography due to groundwater sapping. The configuration chosen is loosely inspired from the early experiments by Jaggar [1908]. Valuable lessons were also drawn from the monograph by Schumm et al. [1987] on experimental fluvial geomorphology. In the present section, the laboratory apparatus, materials, and procedure will first be outlined. Qualitative observations will then be described. Finally, the imaging methods used to obtain quantitative measurements will be presented. In section 7, these measurements will be used to test the theory and computations.

The experiments were conducted in a small dark room at the Hydrotech Research Institute of the National Taiwan University. The physical processes of interest take place in a tilted acrylic sandbox resting on a steel frame (Fig. 11). The central compartment of the sandbox, to be filled with loose sand, has the following inner dimensions: length = 80 cm, width 70 cm, depth = 4 cm. Sand paper is glued onto the floor in order to prevent basal sliding. The compartment is bounded by rigid walls on both sides and by porous barriers upstream and downstream. These barriers are formed by fibre padding held by perforated metal plates. A water recirculation system feeds the apparatus with a steady water discharge. The sand surface evolution is monitored using a video camera and a laser sheet mounted on a motor-driven traverse.

Coarse round sand of median diameter $D_{50} = 0.6$ mm is used for the experiments. The sand is black in color, and responds well to laser illumination. Its measured hydraulic conductivity (at loose-poured packing) is $K = 0.33$ cm/s, obtained from the average of 9 constant head permeameter tests. Fluorescent-dyed water was used to observe capillary rise in sand cones,
yielding for the capillary height an approximate value $h_c \approx 3$ cm. In the experiments reported below, differential sand surface elevations $\Delta z$ never exceed half that value. The inequality $\Delta z < h_c$ is therefore always met, and capillarity-confined conditions prevail. The sand material properties are summarized in table 1.

At the beginning of an experiment, loose wet sand is poured into the box and profiled flush with the lateral walls of the sandbox. The procedure is performed carefully to obtain a flat surface without compacting the sand. Groundwater is then introduced into the system by feeding water from a constant head tank into an upstream transverse ditch. The water in the ditch percolates through the upstream porous barrier into the sand compartment, flows out through the downstream porous barrier, and is collected at the outlet. The flow rate is carefully and gradually increased (over a span of 5-10 minutes) until the sandbox is fully saturated and a very thin film of surface water is observed over most of the exposed sand area. At this stage, the phreatic surface essentially coincides with the inclined sand surface, to the exception of narrow bands next to the upstream and downstream porous barriers. The flow rate is then kept steady, and the process is left to unfold on its own.

The present experiments therefore differ from previous experiments by Howard and McLane [1988], Howard [1988] and Lobkovsky et al. [2004] in two respects. First, our experiments are carried out under constant discharge rather than constant head conditions. Secondly, we use an inclined, constant depth sandbox, whereas the experiments of the above researchers used a sandbox of variable depth (with a horizontal bottom and an inclined sand surface). As a result, we obtain an approximately parallel base flow with no preferential seepage zone along the top surface, as opposed to the localized seepage faces induced in those previous experiments. Probably for this reason, we do not observe immediate erosion once the flow conditions are set. Instead, the thin film condition may last for some time (approx. 5-10 minutes) before any
A perceptible change in the sand surface is observed.

Incipient sand surface evolution takes the form of tiny rills appearing at various locations. Some of them cease to develop after being active for some time, while others gradually grow. Once one or more of these rills grow to sufficient sizes, becoming gullies of the order of 5 mm in width, they are observed to accelerate their evolution. The dominant gullies rapidly adopt well-defined structures similar to those documented by Malin and Edgett [2000] and Schörghofer et al. [2004]. Using the terminology of Malin and Edgett [2000], the gullies exhibit erosional alcoves at their upstream ends, elongated channels originating from the alcoves, and fan-like depositional aprons downstream. Sand motion is apparent along the bottom of the actively flowing channels. These structures and their development are illustrated on Fig. 12. A striking feature of the channels observed in the present experiments is that their discharge may re-infiltrate into the ground before reaching the downstream boundary of the box. This occurs at laterally unconfined channel terminations which appear to be small-scale analogues of the fluvial floodouts described by Tooth [1999].

Once formed, the alcoves, channels and aprons continuously evolve. The alcoves gradually expand outward and migrate headward, accompanied by intermittent slumping of their upstream scarps. The channels tend to adopt oblique paths along the sand surface, and undergo multiple lateral shifts. This leaves a number of abandoned channels on the surface, some of which are later reactivated. Fed by the shifting channels, the depositional aprons build up downstream of the alcoves. Finally, certain depressions may evolve into secondary alcoves, and channels may dissect a deposition fan into primary and secondary aprons. Eventually, one of the alcoves migrates up to the upstream edge of the box and the experiment ends. Some 10 to 15 minutes typically elapse from the onset of rill incision to the end of a run.
In order to observe the gradual sapping process described above, both the flow rate and the sandbox inclination must be carefully chosen. If the discharge is too small, the water will fail to seep out through the sand surface. If on the other hand the discharge is too large, overflow occurs upstream, leading to breaching failure and rapid erosion. Likewise if the slope is too steep, slumping occurs, whereas no channels develop if the slope is too mild. A large number of exploratory experiments was needed to determine appropriate sandbox inclinations and discharges. The inclination $\beta = 14$ degrees was found most favorable, and was adopted for all experiments reported in the present paper. This agrees with the observations of Schörghofer et al. [2004]: in experiments with glass ballotini, these researchers found that channelization due to groundwater seepage was limited to inclinations in the range of 10 to 16 degrees.

At the chosen inclination $\beta = 14$ degrees, the water discharge corresponding to thin film formation and incipient groundwater sapping in the sandbox was measured to be $Q \approx 21$ ml/s. This can be compared with the theoretical discharge needed to obtain uniform parallel flow in an ideal sandbox of the same cross-sectional area $A$:

$$Q_0 = KA\sin\beta.$$  \hspace{1cm} (101)

Setting $A = 70 \text{ cm} \times 4 \text{ cm}$ and using the measured permeability $K = 0.33 \text{ cm/s}$ (see Table 1), the resulting value is $Q_0 = 22.4 \text{ ml/s}$. As $Q \approx Q_0$, a state of uniform flow over the full layer thickness is likely to be a good approximation of the undisturbed groundwater conditions in the sandbox. This of course does not hold near the upstream and downstream boundaries. In particular, the downstream water table is depressed by the free outflow condition along the outside face of the porous barrier. Since we are primarily interested in the surface response away from lateral boundaries (rigid or porous), measurements are focused on a region of interest (shown as a shaded relief on Fig. 11) which does not include these boundary zones.
Imaging methods are used to process the video footage, with the purpose of extracting two types of information. First, images of the scanned laser stripe are used to reconstruct the sand topography at successive stages of the surface evolution. Secondly, ambient light images of the sand surface in the intervals between successive scans are used to monitor the evolving surface runoff pattern. Because they are acquired under the same calibrated camera viewpoint, the topography and runoff patterns can be co-registered in the same 3D frame of reference.

The video camera used is a Sony DV camera operating in interlaced mode at 30 Hz and having a resolution of 480 by 640 pixels. It is mounted rigidly on a tripod and views the evolving sand surface under an oblique angle, as illustrated on Fig. 11. Prior to supplying water to the sandbox, a target of known dimensions is placed onto the undisturbed sand surface and a snapshot of the target is acquired. Pairs of points of known 3D world positions (marked on the target) and known 2D pixel coordinates (pinpointed on the digital image) are then used to calibrate the camera viewpoint using the approach of Spinewine et al. [2003]. As illustrated on Fig. 11, the camera viewpoint is defined by a projection center (apex of the viewing pyramid) and an image plane (base of the pyramid), both positioned in 3D space. Once this viewpoint has been calibrated, a feature point identified on a video image is known to lie on a ray passing through the projection centre and piercing the image plane at the given pixel coordinates. To determine the position of the physical point along this known ray, extra information must be provided, either by using a second camera [e.g., Chandler et al., 2001] or by using structured illumination [e.g. Besl, 1988; Spinewine et al., 2004].

Structured illumination is provided in the present experiments by the laser light sheet. The laser used is a Latronix 12 mW diode laser fitted with light sheet optics, generating a thin red light sheet at an angle of 40 degrees. The laser is mounted on the motor-driven traverse placed
above the sandbox, and aimed vertically downwards. The laser sheet projects a bright transversal stripe onto the sand surface. Viewed at an oblique angle, the stripe delineates the ground topography along a given transect. Its pixel coordinates on each video frame are retrieved automatically by pinpointing brightness maxima. The 3D position of any point along the stripe can then be retrieved as the intersection between the ray issued from the camera viewpoint and the plane spanned by the light sheet. The algorithms needed to perform these various operations are detailed in Ni [2005]. By translating the laser longitudinally using the motor-driven traverse, complete scans of the ground topography can be assembled from a sequence of video images. Scans can further be repeated at regular intervals to capture the gradual evolution of the topography. Because water depths are small even in the most active runoff channels, refraction effects at the air-water interface are not expected to significantly interfere.

Using the above tools, comprehensive sequences of topography maps were obtained for three experimental runs, labelled A, B, and C. The three runs correspond to identical inclination and flow rate conditions. Because the slight irregularities of the initial sand topography differ from run to run, however, contrasted patterns of surface reworking and overland flow were observed. One striking difference is that the three runs A, B, and C evolved one, two, and three primary alcoves, respectively. To illustrate the method, sample topography measurements from an early stage of run A are shown on Fig. 13.

The method presents a number of advantages. First, it is fast and non-intrusive, allowing measurements to be acquired as the process unfolds, without interruption or disruption. Using de-interlaced images at a frequency of 60 Hz, a single scan of 300 transects can be acquired in only 5 seconds. For the experiments reported below, scans were acquired once every 20 seconds for runs A and C, and once every 27 seconds for run B. The resolution and accuracy
that can be attained are also rather good. For each scan, more than 20,000 data points are acquired, for a density of approximately 4 points per centimeter squared. Using Delaunay triangulation, these raw data are interpolated onto a regular grid of sides $\Delta x = \Delta y = 5$ mm.

For each scan, the resulting digital terrain model (DTM) covers an area of interest of at least 40 cm by 50 cm. Validation tests with a surface of known shape were conducted, yielding for the processed DTM a standard error on elevation of approximately 1 mm. This compares favorably with techniques used by previous researchers, as reviewed by Lague et al. [2003]. Details of the validation tests are presented in Ni [2005].

Various means can be used to depict the measured topography $z^{(i)}(x,y)$ reconstructed from each scan. The two representations found to be most useful are illustrated on Fig. 13. On the left panel, contours of height $h^{(i)}(x,y) = -x \sin \beta + z^{(i)} \cos \beta$ (measured with respect to a horizontal datum) are overlain onto a gray-shaded map of elevations $z^{(i)}(x,y)$ (measured with respect to the undisturbed tilted plane). This representation is well-suited to assess quantitatively the spatial distribution of surface elevations. On the right panel, the same topography is represented as a false shaded relief, resulting from the placement of an imaginary grazing light source to the right of the sandbox. This representation is better at resolving fine features of the topography, including small rills formed in otherwise flat zones and narrow channels etched within the erosional alcoves.

To complement the topography measurements, video images are also used to delineate the extent of surface runoff. Under ambient light, running water scintillates on the video footage, allowing visual appraisal of the zones of overland flow. A difficulty is that the boundaries of these zones shift noticeably, even over the short 10-15 second intervals separating successive topography scans. The following semi-automatic procedure was adopted to map the evolving domain covered by overland flow and account for its short term variations. To capture the
runoff pattern associated with each topography map, four short video segments are first extracted from the video footage. One segment is extracted from the half-interval preceding the scan, two segments are extracted from the video acquired during the laser scan, and one segment is extracted from the half-interval coming after the scan.

Each segment has a duration of approximately 5-6 seconds, and is processed in the following way. A standard deviation image is first constructed from the video frames of each segment, highlighting the zones of surface flow. This image is then projected onto the known ground topography using the calibrated camera viewpoint. The pixel image is thus converted to a map in world coordinates that is co-registered with the ground topography on the same DTM grid. This provides a rough map of the instantaneous runoff pattern, illustrated on the left panel of Fig. 14. Zones of surface runoff are then recorded semi-manually (by mouse-clicks), using this map cross-checked against the original video footage. For each segment \( k \), an occupancy index \( I_k(x, y) \) is obtained, set to 1 where runoff is ascertained and to 0 elsewhere.

In order to capture the short-term variability of the runoff pattern, information from each group of four video segments is pooled into a single occupancy map, given by

\[
I(x, y) = \frac{1}{4} \sum_{k=1}^{4} I_k(x, y). \tag{#}
\]

Values of the index \( I \) are equal to 0 where no runoff has been observed on any of the segments, and equal to 1 where runoff has been identified on all four segments, with fractional values 1/4, 1/2, and 3/4 in between. A typical occupancy map resulting from this procedure is shown on the right panel of Fig. 14.
7. Results and comparison

The experiments introduced in the previous section can now be used to test our linear complementarity theory. Each experimental run provides a sequence of topography maps $z^{(s)}(x,y)$ and runoff occupancy maps $I(x,y)$, co-registered in time and space. The topography maps provide the input data necessary for the calculations, while the runoff occupancy maps provide independent output data that can be used to evaluate the predictions. The complete sequence of topography maps acquired for run B is presented on Fig. 15.

For each run, the reference plane $z = 0$ is adjusted to the initial sand surface using a mean elevation fit, and is assumed to define the undisturbed water table throughout the experiment. Each topography map $z^{(s)}(x,y)$ is then used as input for coupled groundwater-overland flow calculations. Since steady flow is assumed, calculations for different maps are independent of each other. Unless specified otherwise, the capillarity-confined, finite depth version of the theory (section 4) is adopted. The rigid floor of the sandbox is represented by an impervious plane parallel to the reference plane at depth $z = -D$ where $D = 4$ cm. The assumed undisturbed base flow is distributed uniformly over this depth with specific discharge $U_s = K \sin \beta$ controlled by the sandbox inclination $\beta = 14$ degrees. Lateral boundaries are ignored. Under the assumptions of the theory, these data provide all the information needed to calculate the perturbed groundwater flow and associated surface drainage pattern.

The predicted sequence of overland flow patterns is presented on Fig. 16. A network representation is chosen, with the width of each link set proportional to the conveyed discharge $Q$. The computations shown use the measurement grid at its full resolution ($5$ mm × $5$ mm). As a check, calculations (not shown) were also performed for a coarse-grained grid of
half the resolution (10 mm × 10 mm). Despite some loss of detail, they were found to yield similar drainage patterns, indicating a reasonable degree of mesh independence. In order to evaluate the predictions, independent experimental evidence is needed. This is provided by the sequence of observed runoff maps shown on Fig. 17. Each runoff occupancy map is contemporaneous with the corresponding topography map, hence contemporaneous also with the drainage pattern predicted from the topography data.

The evolution depicted on Figures 15 to 17 underscores the challenges associated with seepage-driven geomorphic flows. The drainage pattern takes the form of a constantly evolving network, where channels can be activated and deactivated repeatedly in the course of a single episode. For pure overland runoff conveying accumulated rainfall out of a watershed, drainage networks connect their branches into a single continuous tree. Coupled with groundwater motion, by contrast, drainage networks can become disconnected. Linked to a shallow aquifer, channels can initiate and terminate abruptly, and distinct branches of the surface network can be severed from each other. As illustrated on Fig. 15 to 17, network disconnections and reconnections can and do occur as a result. It is also clear from the figures that the topography evolution is itself driven by the groundwater seepage and surface runoff.

A complete theory of the process would thus need to model as well the sediment motion and geomorphic action of the flowing water. This geomorphic response lies outside the scope of the present theory, which instead deals exclusively with topographic control of the coupled groundwater-overland flow.

Before comparing calculations and observations, a rough chronology of experimental run B can be reconstructed from the three series of 20 successive panels, numbered B1 to B20. Five successive stages of development can be identified. In the first stage (B1-B3) two preferential springs emerge from a background of distributed seeps associated with uncontrolled
irregularities of the initial topography. In the second stage (B4-B7), sapping by these two springs leads to the rapid development of two primary alcoves (labeled P1 and P2). Runoff from these two depressions is conveyed downstream by narrow channels which tend to follow oblique paths along the sandbox surface. Concurrently, the isolated seeps distributed elsewhere along the surface dry up.

In the third stage (B8-B10), streamflow withers inside the smaller downstream alcove P2, pausing its erosional development. Meanwhile, the larger upstream alcove P1 continues to expand and migrate headward. In the fourth stage (B11-B13), alcove P2 is rejuvenated, starting to develop again under the influence of renewed seepage. Between scans B13 and B14, runoff from alcove P1 is suddenly rerouted to alcove P2. In the ensuing fifth stage (B14-B18), the combined seepage from active alcoves P1 and P2 flows through the channel system of alcove P2. Once disconnected, the right-hand channel associated earlier with alcove P1 gradually drains out from upstream to downstream. Only the stem of this abandoned channel remains active, forming an incipient secondary alcove. Water supply to the sandbox is turned off between scans B18 and B19.

Shown respectively on Figures 16 and 17, the predicted and observed runoff patterns associated with these developments are in fair agreement with each other. The five stages described above can be retraced from either figure, which depict qualitatively similar pattern evolutions. The calculations have no problem in dealing with the partly disconnected nature of the drainage pattern. Nevertheless, certain temporal and spatial discrepancies must be noted. First, the timing of certain transitions is off. In particular, the channel avulsion which reroutes runoff from alcove P1 to alcove P2 is captured by the calculations, but its occurrence is delayed by some 80 seconds (scan B17 instead of scan B14). Because the computations lag behind the observations, this time shift is not believed to be due to unsteady effects. Also, no
lag is recorded for various other features. The timing of the deactivation (scan B8) and reactivation (scan B11) of alcove P2, for instance, is correctly predicted.

Secondly, some spatial features of the drainage pattern are missed or misplaced. In the first stage of development, for example, the distribution of seeps is not well depicted. A likely reason is that the irregularities of the topography at this stage are of the same order (~1 mm) as the accuracy of the digital terrain model. As rills become more deeply incised, their resolution on the topographic maps improves. Encouragingly, even if the distributed seeps are not accurately positioned, their existence during the first stage and gradual disappearance during the second stage are captured qualitatively by the calculations.

Other discrepancies are inherent to some of the assumptions underlying the theoretical calculations. In the model, for instance, the runoff discharge (Fig. 16) is conveyed by channels of zero water depth and vanishing width. In the actual experiments (Fig. 17), water flows through channels of finite wetted perimeter. Likewise, channels which terminate along the sand surface end at a single point in the model. In the experiments, the channel-terminating infiltration tends to occur as non-channelized sheetwash. On the positive side of the ledger, zones of diffuse seepage associated with the alcove depressions are portrayed rather realistically as dense drainage networks nested within the alcoves. Unexpectedly, another favorable feature of the model can be noted. Although at first sight the adopted surface routing method does not allow divergent channel branching, bifurcations into multi-threaded channels do occur in the calculations. Scan B10 features a conspicuous example, where calculations predict rather accurately the observed two-threaded channel flow. Divergent bifurcations can occur in the present description because two adjacent, parallel surface channels can exchange discharge with each other through groundwater flow.
To further illustrate the runoff response associated with various surface topographies, selected snapshots from runs A, B, and C are presented on Fig. 18. As mentioned above, the most salient contrast between the three runs lies in the number and configuration of eroded alcoves. A single primary alcove P1 developed during run A, with a secondary alcove S1 formed at later stages (see Fig. 18a-c). For run B, the intermittent evolution of two primary alcoves P1 and P2 was described earlier (Figures 15 to 17). For run C, three primary alcoves P1, P2, P3 and two secondary alcoves S1, S2 evolved in a roughly staggered configuration (see Fig. 18g-i). In order to provide a slightly different perspective, the laser-scanned topography measurements are represented as contour maps on the left panels (a,d,g) of Fig. 18. On the middle panels (b,e,h), the predicted surface seepage is shown as a gray level map, with dark tones indicating effluent seepage (groundwater drainage) and bright tones indicating influent seepage (recharge). The predicted runoff coincides with zones of active seepage. On the right panels (c,f,i), the observed runoff patterns are presented for comparison.

Again, the predicted and observed runoff patterns are in fair agreement with each other. Agreement is poorest for the top snapshots, taken from an early stage of development of run A. Zones of groundwater drainage are reasonably well-captured, but the zones of recharge are wrongly positioned. The calculated runoff tends to be routed straight downslope, while the observed runoff adopts oblique paths. This is likely due in part to errors of the topography measurements, which are of the same order as the actual topography variations (~ 1 mm) for weakly incised channels. A single erroneously lowered pixel in a downstream levée confining an oblique channel is all it takes for the predicted flow to breach through and head straight downstream. Such errors in the routing of the surface flow affect the entire downstream path of the runoff. Taken at a later stage of development, the channel network is more strongly incised for the snapshots of run B and C, and better agreement is registered. It is also instructive to interpret the right panels (c,f,i) of Fig. 18 (observed runoff maps) with the help
of the middle panels (b,e,h) of the same figure (predicted seepage distribution). Gaining channels (supplied by effluent seepage $s > 0$) are found to be rather stable (black zones on the right panels indicating steady occupancy). By contrast losing channels (depleted by influent seepage $s < 0$) tend to migrate over short timescales (gray zones indicating shifting occupancy).

One may also wonder whether the observed seepage distribution and zones of overland flow could not be approximated using simpler methods. Figure 18 provides some answers to this question. First, it is apparent that the zones of effluent seepage (middle panels of Fig. 18) roughly coincide with the more deeply incised zones of the sand surface (left panels). In our experiments, the incision depth is equivalent to a height below the unperturbed water table, and this would naturally favor effluent seepage. Nevertheless, the correlation is limited, and the relation between incision depth and rate of effluent seepage is not one-to-one. Points of the surface sharing the same incision depth can be characterized by effluent seepage, influent seepage or no seepage at all depending on their location with respect to the drainage network. This indicates that local point-wise correlations will fail, and that a global coupled treatment of the kind carried out here is necessary to map the zones of seepage and flow.

To the extent that the calculations can be trusted, they provide of course more information than just the pattern of surface runoff and seepage. The underlying groundwater flow pattern is also obtained as part of the solution. This is illustrated on Figures 19 and 20. Figure 19 shows a 3D block diagram of the surface and subsurface flows predicted for scan C46. Three transverse cross-sections through the same flow field are shown on Fig. 20. On Fig. 19, the water table and groundwater streamlines along the aquifer surface bear the imprint of the overlying channel network. In addition to the influence of surface drainage emphasized by previous researchers [Jaggar, 1908; Schörghofer et al., 2004], the present results exhibit some
clear effects due to recharge and capillary confinement. At zones of effluent seepage such as the alcoves, the water table is depressed and streamlines converge. Conversely, elevated ridges of the water table and divergent streamlines are observed along losing channels, where influent seepage replenishes the aquifer. Independently of the seepage distribution, the sand surface exerts a purely topographic influence on the groundwater flow due to capillary confinement. In particular, the upstream arc of the alcoves acts as a footprint boundary which forces an outward deflection of the streamlines. Further downstream, these same streamlines are pulled inwards by zones of effluent seepage nested within the alcoves.

Equipotentials and groundwater flux vectors are plotted on Fig. 20 for three transverse cross-sections cut across the 3D flow field. These slices are roughly representative of the upstream, middle, and downstream regions of the sandbox. Drainage or recharge occur wherever the sand surface and groundwater table intersect. Slice a is cut through the upstream-most alcove P1, where strong effluent seepage is observed (symbolized by the upward arrow). Little transverse motion is observed away from this surface sink. Slice b is cut through the middle of the domain. Effluent seepage is observed at the other two alcoves P2 and P3 located to the left. To the right, runoff from alcove P1 re-infiltrates as localized recharge from a losing channel. Referring back to Fig. 19, this surface channel terminates further downstream after losing all its discharge. One can see from cross-section 20b that the lost discharge does not merely diffuse into the aquifer. Instead, a substantial flux flows laterally to alcove P3, amounting to a groundwater bypass from the channel to the alcove. This emphasizes the fact that the disjoint surface network is in fact connected by sub-surface throughflow. Finally, slice c is cut through the downstream part of the domain. Groundwater seepage in this zone is dominated by recharge (from two surface channels). Topography effects are also apparent: irregularities in terrain inclination lead to localized upflow and downflow in zones where no seepage occurs.
A view of the sandbox surface at the end of run C is shown on Fig. 21. Finely incised grooves etched by the groundwater-fed streamflow are apparent on the photograph. These fine-scaled features are not resolved by the laser scans. Nevertheless, the zones where these grooves are encountered roughly coincide with the domain of activated surface channels predicted by the calculations (see Fig. 19). In particular, it can be noted that zones of channelized streamflow do not cover the full extent of the alcove depressions. Non-channelized heels of the alcove footprints are observed on both the photo of Fig. 21 and the calculations of Fig. 19. Gravity slumping rather than streamflow erosion is surmised to be the active geomorphic mechanism near the upstream scarp of the alcoves. Such joint action by mass wasting and fluvial transport was documented in detail in the sapping experiments of Howard and McLane [1988].

Based on the above results, we can motivate and evaluate a posteriori the assumptions underlying the proposed theory. First, the perturbed surface exhibits a range of horizontal scales, including both narrow and wide features. The channels incised by the streamflow have typical widths $w$ of 1 cm and below. On the other hand, the depressed alcoves have typical widths $W$ of up to 10 cm at late stages of their development. Both dimensions can be gauged relative to the depth of the sandbox $D = 4$ cm. Small ratios $w/D$ confirm that a vertically averaged description based on the Dupuit assumption would not be appropriate here, justifying our choice of a three-dimensional approach. The three-dimensional character of substream flow paths was earlier emphasized by Harvey and Bencala [1993]. On the other hand, large ratios $W/D$ suggest that aquifer confinement by the impervious floor is a significant factor. The transverse slices of Fig. 20 illustrate the influence of the bottom on the flow. In this regard, the patterns are rather similar to those calculated by Anderson [2003] for surface driven groundwater flow in two-dimensional strips of finite depth.
Overall, the linearized treatment adopted in the present work appears justified. The vertical scale $\Delta z$ of the surface perturbations remains small to moderate, up to 1 cm at most. Thus typical inclinations $\Delta z/W$ in zones of effluent seepage associated with the alcove regions are consistent with the small slope approximation. Nevertheless, the approximation does not hold everywhere. It clearly becomes questionable near ridges of localized recharge associated with narrow losing channels, where the inclination $\Delta z/W_z$ is of order 1. It is likely that this limitation of the linearized treatment partly explains why the calculations agree more closely with the observations in zones of groundwater drainage than in zones of recharge.

In order to evaluate the relative influence of bottom and surface confinement, Figure 22 contrasts results obtained under different sets of assumptions. Panels a and b show calculations under capillarity-confined conditions at the surface, subject also to bottom confinement by an impervious plane at depth $z = -D = -4$ cm. These are the assumptions expected to match most closely the actual conditions in the experimental sandbox, and they have been adopted for all the calculations discussed above. Panels c and d show calculations under capillarity-confined conditions at the surface, but with an aquifer of infinite depth $D \to \infty$ below. Finally, panels e and f show calculations for a capillarity-free aquifer of infinite depth, using the theory derived in section 3.

Although results for these three scenarios are qualitatively similar, a number of differences can be noted. The most significant changes are associated with the switch from finite to infinite depth. Quantitatively, the maximum surface discharge jumps from 1.3 ml/sec to 5 ml/sec upon sending the depth $D$ to infinity. Compared to the finite depth calculations of panels a and b, the active drainage network is also found to grow upstream and downstream (panels c-f). Upstream, dense drainage patterns now occupy nearly the full extent of the alcove depressions. Downstream, the channel issued from primary alcove P3 now reaches the
edge of the domain instead of terminating over the apron. In both respects, the finite depth calculations are in better agreement with the runoff pattern observed in the experiments (see Fig. 18).

By contrast, the differences registered when switching from capillarity-confined to capillarity-free boundary conditions at the surface are relatively minor. The drainage patterns observed on panels c and e, respectively, are nearly identical. The only notable difference involves the streamlines of panels d and f. Whereas the capillarity-confined flow of panel d is influenced by topography even in the absence of seepage, the capillarity-free flow of panel f is controlled by the seepage distribution alone. This is best seen upstream of the alcoves, where streamlines are deflected outwards then inwards in the capillarity-confined case. In the capillary-free case, only the inward pull due to seepage is felt. Nevertheless, this effect is too small to measurably affect the overland flow pattern. It was noted earlier that in the linearized approximation, capillarity-free and capillarity-confined behaviors differ only due to interactions between the perturbations and the inclined base flow. In the present case, these interactions are limited because of the moderate slope of the sandbox ($\beta = 14$ degrees). One could in fact neglect them altogether and simply superpose the unperturbed base flow with drainage-induced perturbations evaluated as if there was no base flow. In particular, one could use fundamental solution (56) in the capillarity-free description (42), and neglect term $z_0^{(w)}$ in the capillarity-confined equation (61). For greater inclinations, however, more pronounced differences can be expected and such simplifications may lead to large errors.

The limited influence of capillarity in the present case has some important implications for laboratory work. Because of capillarity rise, it is commonly assumed [Bear, 1972] that small scale sandbox experiments are ill-suited to model large scale flows involving free surfaces. While this may hold true in general, the present work suggests that exceptions to the rule can
be carved out. Specifically, our calculations indicate that when topography perturbations remain within the capillary fringe, capillarity effects exert very little influence on drainage patterns. To study interactions between channel networks and shallow aquifers, small scale capillarity-confined experiments can therefore be accepted as reasonable approximations of large scale capillarity-free systems. Paradoxically, bigger is not necessarily better for such tests, because one must ensure that variations do not exceed the capillarity fringe. This confirms in retrospect the intuitions of Jaggar [1908], who used a sandbox of 45 cm by 60 cm to set up his pioneering seepage stream experiments. This of course does not mean that capillarity will not affect in other ways the geomorphic evolution of small-scale systems. For instance, the effective cohesion that results from capillary tension is expected to exert a significant control on the slumping of the alcove headscarps [see Howard and McLane, 1988].
8. Conclusions

In the present work, a combination of theory, computation and experiment was used to probe interactions between surface channels and shallow aquifers. The proposed theory treats surface drainage and recharge as perturbations of a parallel base flow. This was shown to lead to a simple integral equation linking groundwater motion to overland runoff. To delineate the domain of active overland flow, simultaneous constraints on the water table and surface runoff were imposed. The resulting system of equations and inequalities was cast as a linear complementarity problem (LCP). This reduces the coupling between groundwater and surface flow to a surprisingly compact formulation. Variants of the theory applicable to capillarity-free and capillarity-confined conditions along the surface were found to share the same essential structure. Using the method of fundamental solution, the formulation was further discretized into an algebraic LCP. By exploiting the special structure of the problem, it was found possible to avoid assembling large matrices, allowing numerical solutions on fine grids to be computed at little cost.

To test the approach, computations were compared to detailed experimental data obtained from small-scale laboratory tests. The experiments involve a tilted sandbox, with a surface that undergoes channelization due to groundwater seepage. Imaging methods were used to obtain topography and runoff maps, co-registered in space and time. Using the measured baseflow and topography data, linear complementarity calculations were used to predict the surface and groundwater flow. The calculated patterns of overland flow were found to be in fair agreement with the runoff observations. Encouragingly, this level of agreement was obtained without any tuning parameter, and despite limitations in the accuracy of the topography data. To the best of our knowledge, these results represent the first successful predictions of runoff patterns for disconnected networks of irregular channels fed by groundwater seepage. Beyond predicting runoff patterns, the calculations provide quantitative
estimates of the surface and groundwater fluxes. They also permit visualization of the invisible pattern of groundwater motion beneath the sand surface. This was used to highlight the three-dimensional character of the subsurface fluxes, the joint influence of drainage and recharge, and the effects of bottom and surface confinement on the groundwater flow.

Nevertheless, various limitations of the present study must again be underlined. First, the adopted linearized treatment is limited to situations where deviations of the water table away from a reference plane can be considered small. Aquifers and ground surfaces with more complicated geometries are thus excluded. The additional restriction to either capillarity-free or capillarity-confined aquifers represents another significant limitation of the present approach. The theater-headed canyons observed by Otvos [1999] on sand beaches, for instance, feature topography variations of intermediate scale which likely exceed the capillary height, without being so large that the capillarity height can be neglected. The details of the capillary fringe may thus significantly affect the water (and sand) response, and require a more complicated treatment. Further adjustments should also be made to take water ponding into account. This would be needed to describe groundwater systems linked to both lakes and streams, an important issue in practice [Winter, 1999]. Finally, the proposed theoretical and computational approach dealt exclusively with the water motion, without attempting to predict its geomorphic effects. This would of course be necessary to predict rather than monitor the evolving topography of the small-scale experiments used to validate the approach. Further research efforts will be needed to address these limitations.
Appendix: derivation of steady Green functions for the capillarity-free aquifer

To solve equations (30)-(32), it is useful to examine first the unsteady version of the same problem. In dimensional variables, the linearized equations governing unsteady, unconfined groundwater flow can be written

\[
\text{Laplace: } \nabla^2 \phi = 0, \quad z < 0, \quad (102)
\]

Kinematic surface BC:

\[
\frac{N}{\cos \theta} \frac{\partial \phi}{\partial t} + K \tan \beta \frac{\partial \phi}{\partial x} + K \frac{\partial \phi}{\partial z} = r, \quad z = 0, \quad (103)
\]

Dynamic surface BC:

\[
\phi = z^{(w)} \cos \beta, \quad z = 0, \quad (104)
\]

Initial conditions:

\[
z^{(w)} = z_0^{(w)}, \quad t = 0, \quad (105)
\]

where \(N\) is the porosity of the aquifer. We use dimensional variables in this appendix because dimensional analysis will prove helpful for an important step of the derivation. The time-evolving potential \(\phi(x, y, z, t)\) and water table \(z^{(w)}(x, y, t)\) can be expressed in terms of unsteady Green functions \(\hat{\phi}(x, y, z, t)\), \(\hat{z}^{(w)}(x, y, t)\). These represent the aquifer response to an impulsive recharge

\[
r(x, y, t) = \hat{V} \delta(x, y, t), \quad (106)
\]

where \(\hat{V}\) is a unit volume, introduced to preserve dimensional homogeneity. The same response can be obtained by setting the recharge distribution to zero and starting from initial conditions \(z_0^{(w)}(x, y) = \hat{V} \delta(x, y)\). We will follow this alternative route and seek solutions to the following homogeneous initial value problem

\[
\text{Laplace: } \nabla^2 \hat{\phi} = 0, \quad z < 0, \quad (107)
\]

Kinematic surface BC:

\[
\frac{N}{\cos \theta} \frac{\partial \hat{\phi}}{\partial t} + K \tan \beta \frac{\partial \hat{\phi}}{\partial x} + K \frac{\partial \hat{\phi}}{\partial z} = 0, \quad z = 0, \quad (108)
\]

Dynamic surface BC:

\[
\hat{\phi} = \hat{z}^{(w)} \cos \beta, \quad z = 0, \quad (109)
\]

Initial conditions:

\[
\hat{z}^{(w)} = \hat{z}_0^{(w)} = \hat{V} \delta(x, y), \quad t = 0. \quad (110)
\]

Symmetries of the system can now be exploited to gradually simplify the set of four
independent variables \((x, y, z, t)\). If we can reduce this set to a single variable, then the PDE becomes a more tractable ODE. The first step is to reduce the three-dimensional problem (in a stationary frame of reference) to a radially symmetric problem (in a moving frame of reference). Define transformed variables as
\[
\tau = t \cos \beta, \quad \xi = x - \frac{K}{N} \sin \beta t.
\] (111)

Then by the chain rule,
\[
\frac{\partial \phi}{\partial \tau} = \cos \beta \frac{\partial \phi}{\partial \tau} - \frac{K}{N} \sin \beta \frac{\partial \phi}{\partial \xi}, \quad \text{and} \quad \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi}.
\] (112)

In the new variables, the Laplace equation and dynamic boundary condition do not change. However the kinematic boundary condition reduces to
\[
N \frac{\partial \phi}{\partial \tau} + K \frac{\partial \phi}{\partial z} = 0,
\] (113)
which has the same form as the condition for a horizontal water table (i.e. taking \(\beta = 0\)). This was first noticed by Dagan [1967a]. For a horizontal water table, the problem is radially symmetric, hence the two variables \((\xi, y)\) can be replaced by radius
\[
\rho = \sqrt{\xi^2 + y^2}.
\] (114)

In the reduced variables \((\rho, z, \tau)\), the equations can be rewritten

Laplace:
\[
\nabla^2 \tilde{\phi} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \tilde{\phi}}{\partial \rho} \right) + \frac{\partial^2 \tilde{\phi}}{\partial z^2} = 0, \quad z \leq 0.
\] (115)

Kinematic surface BC:
\[
N \frac{\partial \tilde{\phi}}{\partial \tau} + K \frac{\partial \tilde{\phi}}{\partial z} = 0, \quad z = 0.
\] (116)

Dynamic surface BC:
\[
\tilde{\phi} = z^{(w)}, \quad z = 0.
\] (117)

Initial condition:
\[
z^{(w)}(\tau) = \tilde{z}^{(w)}(\tau), \quad \tau = 0.
\] (118)

It can next be observed that regardless of the initial condition, solutions to the above homogeneous problem are characterized by the following time-depth symmetry
\[
\tilde{\phi}(\rho, z, \tau + \frac{\rho}{K} \alpha) = \tilde{\phi}(\rho, z - \alpha, \tau)
\] (119)
where \(\alpha\) is an arbitrary elevation offset. This symmetry means that the future potential at a
certain level is equal to the current potential at a lower depth. Differentiating both sides with respect to \( \alpha \), it follows that

\[
N \frac{\partial \hat{\phi}}{\partial \tau} + K \frac{\partial \hat{\phi}}{\partial z} = 0
\]  

which now holds everywhere (not just on the free surface). This implies that

\[
\hat{\phi}(\rho, z, \tau) = \hat{\phi}(\rho, \zeta)
\]  

where \( \zeta = z - \frac{N}{K} \tau \) is a new reduced variable. In the reduced variables \( (\rho, \zeta) \), the kinematic boundary condition is automatically satisfied, and it remains to satisfy the Laplace equation

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \hat{\phi}}{\partial \rho} \right) + \frac{\partial^2 \hat{\phi}}{\partial \zeta^2} = 0,
\]

complemented by condition (which can either be seen as a boundary or as an initial condition)

\[
\hat{\phi}(\rho, z = 0, \tau) = \hat{\phi}(\rho, \zeta = 0) = z_o^{(w)}.
\]

The next step follows from dimensional analysis. Because the response is linear with respect to the injected volume \( \hat{V} \), and because the impulsive initial condition has no length scale of its own, the potential can be written in the functional form

\[
\hat{\phi} = F(\rho, \zeta).
\]

Taking radius \( \rho \) as repeating variable, dimensional analysis then yields

\[
\hat{\phi} = f(\zeta / \rho) \frac{\hat{V}}{\rho^2}.
\]

Our final reduced variable can then be taken as the ratio of depth over radius

\[
\sigma = \zeta / \rho.
\]

In terms of this single reduced variable, the Laplace PDE (122) becomes (after repeated application of the chain rule) the ODE

\[
(\sigma^2 + 1) \frac{d^2 f}{d\sigma^2} + 5\sigma \frac{df}{d\sigma} + 4 f = 0.
\]

This is a second order linear equation with variable coefficients. The solution of interest (vanishing at infinity) is given by
\[ f(\sigma) = C \frac{\sigma}{(\sigma^2 + 1)^{3/2}} \]  

(128)

where \( C \) is a constant yet to be determined. Its value is determined by the condition that the volume of water is conserved at all times

\[ 2\pi \int_0^\infty \hat{z}^{(w)}(\rho, \tau) \rho \, d\rho = 2\pi \int_0^\infty \tilde{z}^{(w)}_0(\rho) \rho \, d\rho = \frac{\hat{V}}{N}. \]  

(129)

Substitution of

\[ \hat{z}^{(w)}(\rho, \tau) = \frac{\hat{V}}{\rho^2} f\left(-\frac{K}{N} \tau / \rho\right) = -C\hat{V} \frac{\frac{K}{N} \tau}{\left\{\left(\frac{K}{N} \tau\right)^2 + \rho^2\right\}^{3/2}} \]  

(130)

yields

\[ C = -\frac{1}{2\pi N}. \]  

(131)

Transforming back from the reduced variables to the original variables, the unsteady fundamental solutions are obtained as

\[ \hat{\phi}(x, y, z, t) = \frac{\hat{V}}{2\pi N} \frac{\frac{K}{N} t - z}{\left\{\left(\frac{K}{N} t - z\right)^2 + (x - \frac{K}{N} \tan \beta t)^2 + y^2\right\}^{3/2}}, \]  

(132)

\[ \hat{z}^{(w)}(x, y, t) = \frac{\hat{V}}{2\pi N \cos \beta} \frac{\frac{K}{N} t}{\left\{\left(\frac{K}{N} t\right)^2 + (x - \frac{K}{N} \tan \theta t)^2 + y^2\right\}^{3/2}}, \]  

(133)

where it is implied that \( t \geq 0 \) and that \( \hat{\phi}(x, y, z, t) = \hat{z}^{(w)}(x, y, t) = 0 \) for \( t < 0 \). These Green functions apply to the transient evolution of an aquifer of infinite depth and infinite lateral extent, with a sloping water table. They were first presented by Dagan [1967a], who derived them using a more general Green function obtained by transform techniques. The approach presented above constitutes an alternative way of deriving Dagan’s results.

To obtain the steady fundamental solutions \( \hat{\phi}(x, y, z) \) and \( \hat{z}^{(w)}(x, y) \), the final step of the derivation is to integrate in time the unsteady fundamental solutions (132) and (133):

\[ \hat{\phi}(x, y, z) = \frac{\hat{R}}{V} \int_{-\infty}^0 \hat{\phi}(x, y, z, -\tau) \, d\tau \]  

(134)

\[ \hat{z}^{(w)}(x, y) = \frac{\hat{R}}{V} \int_{-\infty}^0 \hat{z}^{(w)}(x, y, -\tau) \, d\tau \]  

(135)

where \( \hat{R} \) denotes a unit recharge rate. In other words, the steady Green functions are
obtained as the response to steady sources turned on an infinitely long time ago. Using the unsteady fundamental solutions derived above and the change of variable \( \tau' = -\frac{K}{N} \tau \), the integral for the potential can be rewritten

\[
\hat{\phi}(x, y, z) = \frac{\hat{\mathcal{R}}}{2\pi K} \int_0^{\infty} \frac{\tau' - z}{\{(\tau' - z)^2 + (x - \tan \beta \tau')^2 + y^2\}^{3/2}} \, d\tau'. 
\] (136)

Working out the integral above is lengthy but straightforward, yielding

\[
\hat{\phi}(x, y, z) = \frac{\hat{\mathcal{R}}}{2\pi K} \frac{\tan \beta (x - z \tan \beta)(x^2 + y^2 + z^2)^{1/2} + (x^2 + y^2 - xz \tan \beta)(1 + \tan^2 \beta)^{1/2}}{(x - z \tan \beta)^2 + (1 + \tan^2 \beta)y^2}(1 + \tan^2 \beta)^{1/2}(x^2 + y^2 + z^2)^{1/2}. 
\] (137)

The fundamental solution for the free surface is likewise obtained as

\[
\hat{z}^{(w)}(x, y) = \frac{\hat{\mathcal{R}}}{2\pi K \cos \beta} \frac{x \tan \beta (x^2 + y^2)^{1/2} + (x^2 + y^2)(1 + \tan^2 \beta)^{1/2}}{\{x^2 + (1 + \tan^2 \beta)y^2\}(1 + \tan^2 \beta)^{1/2}(x^2 + y^2)^{1/2}}. 
\] (138)

Expressions (36)-(37) are dimensionless versions of these results.
Acknowledgements

Financial support from the National Science Council and Council of Agriculture, Taiwan, is gratefully acknowledged. Special thanks are also extended to Prof. Young Der-Liang (Natl Taiwan Univ.) for teaching us the method of fundamental solution, to Dr. Benoit Spinewine (Univ. catholique de Louvain, Belgium) for lending us laser and advice during his visit to Taiwan, and to Professors Chen Su-Chin (Natl Chung-Hsing Univ.), Wu Fu-Chun and Lin Ming-Lang (Natl Taiwan Univ.) for providing useful feedback.
References


Lague, D., A. Crave, and P. Davy, Laboratory experiments simulating the geomorphic


Ni, W.J., Groundwater drainage and recharge by geomorphically active gullies, M.S. Thesis, Graduate Institute of Civil Engineering, National Taiwan University, 2005.


Table captions

Table 1. Properties of the sand material used for the laboratory experiments

Figure captions

Figure 1. Small braiding plain drained by a retreating network of surface channels fed by groundwater seepage. Ba Ung Ung Brook, Taiwan East Coast, in the wake of Typhoon Morakot on August 10, 2003 (photo by H. Capart). Scale provided by the circled bystander.

Figure 2. Gravel bed river at low stage, featuring separate surface channels connected by groundwater throughflow. Chung Kang River, Central Taiwan, on February 11, 2005 (photo by H. Capart). Scale provided by the circled group of three bystanders.

Figure 3. Aquifer and surface topography described in a sloping coordinate system.

Figure 4. Boundary conditions near the ground surface: a) capillarity-free aquifer; b) capillarity-confined aquifer.

Figure 5. Superposition of base flow and point recharge of unit strength for capillarity-free aquifer of slope $\beta = 14$ degrees. Top shaded surface is the water table and streamlines depict the groundwater flow.

Figure 6. Surface streamlines of the fundamental solutions for capillarity-free aquifer of slope $\beta = 14$ degrees: a) point recharge fundamental solution $\phi_{ef}$; b) downslope doublet $\partial \phi_{ef} / \partial \xi$; c) transverse doublet $\partial \phi_{ef} / \partial \eta$. For clarity, the base flow component has not been added.
Figure 7. Superposition of base flow and a source-sink pair of unit strength for capillarity-confined aquifer of slope $\beta = 14$ degrees and depth $D = 3L$. Shading denotes pore water pressure and streamlines depict the groundwater flow.

Figure 8. Downslope routing of the overland flow: a) streamlines depicting downslope vector field $j^{(i)}(x, y)$; b) reduction of the vector field to a discrete channel network. The network covers the entire surface but will not be activated in full by groundwater seepage. The topography used is taken from scan B11 (see experiments).

Figure 9. Superposition of base flow and fundamental solutions (FS) along center plane: a) isolated source represented by capillarity-free FS $\hat{\phi}_{cl}$; b) source-sink pair represented using the capillarity-confined FS $\hat{\phi}_{cc}$; c) isolated source represented by regularized capillarity-free FS $\tilde{\phi}_{cl}$; d) source-sink pair represented using the regularized capillarity-confined FS $\tilde{\phi}_{cc}$. The regularization length is set here to $\ell = \frac{1}{10} L$.

Figure 10. Results of forward and inverse computations for uniform recharge over a unit disk: a) analytical recharge distribution; b) analytical water table response [Dagan, 1967a]; c) comparison of forward MFS computations (dots) along $y = 0$ with the radially symmetric analytical response $z^{(w)}(\rho)$ (line); d) comparison of inverse MFS computations (dots) along $y = 0$ with the radially symmetric analytical forcing $r(\rho)$ (line).

Figure 11. Laboratory sandbox instrumented with laser scanning rig and video camera: a) photo of the set-up; b) sketch of the sandbox and imaging configuration.

Figure 12. Observed geomorphic evolution of the sand surface (run A): a)-c) sequence of
oblique photos; d) interpreted sketch.

Figure 13. Laser scanned topography at early stage of run A (scan A13): a) gray shaded map of erosion (dark) and deposition depths (bright) with elevation contours every 2 mm; b) false relief map obtained by illuminating the topography from the right with a grazing light.

Figure 14. Surveyed overland flow at early stage of run A (scan A13): a) standard deviation image measuring the scintillations associated with flowing water on a short video segment; b) overland flow occupancy map obtained from four such video segments.

Figure 15. Ground topography evolution for run B. False relief maps show the topography measured by successive laser scans. Scans are acquired every 27 seconds.

Figure 16. Predicted overland flow network for run B, calculated from the topography maps of Fig. 15. The width of each network link is set proportional to the conveyed discharge $Q$.

Figure 17. Observed surface runoff pattern for run B. These overland flow occupancy maps are co-registered with the topography maps of Fig. 15.

Figure 18. Synoptic view of the measured topography (a,d,g), predicted seepage rate (b,e,h), and observed surface runoff (c,f,i) for snapshots of three experimental runs: a-c) scan A12; d-f) scan B11; g-i) scan C46. Letters P and S denote primary and secondary alcoves. Negative seepage rates indicate recharge of the aquifer.

Figure 19. Block diagram showing the measured ground topography and calculated overland
flow network (above), as well as the calculated phreatic surface and groundwater flow streamlines (below). The topography is from scan C46.

Figure 20. Transverse cross-sections through the block diagram of Fig. 19: a) upstream cross-section; b) middle cross-section; c) downstream cross-section. Thin lines = equipotentials; vectors = groundwater fluxes.

Figure 21. Photograph of the sandbox surface at the end of run C (after water has been drained out).

Figure 22. Calculations of the coupled groundwater-overland flow pattern under various assumptions: a,b) capillarity-confined flow and aquifer of finite depth $D = 4$ cm; c,d) capillarity-confined flow and aquifer of infinite depth $D \to \infty$; e,f) capillarity-free flow and aquifer of infinite depth $D \to \infty$. Left panels (a,c,e) show the topography and calculated overland flow network; right panels (b,d,f) show the water table and streamlines along the aquifer surface. The topography is from scan C46.
## Tables

<table>
<thead>
<tr>
<th>Property</th>
<th>Measured value</th>
<th>Method of determination</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median diameter $D_{50}$</td>
<td>0.6 mm</td>
<td>Sieve analysis</td>
</tr>
<tr>
<td>Hydraulic conductivity</td>
<td>0.33 cm/s</td>
<td>Mean of 9 constant head permeameter tests</td>
</tr>
<tr>
<td>(permeability) $K$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capillary height $h_c$</td>
<td>$\approx 3$ cm</td>
<td>Observed capillary rise of fluorescent-dyed water</td>
</tr>
</tbody>
</table>

Table 1
Figures

Figure 1

Figure 2
Figure 3

Figure 4
Figure 7

Figure 8
Figure 9
Figure 10

Figure 11
Figure 12
Figure 15
Figure 16
Figure 17
Figure 18
Figure 22